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On the Asymptotic Behavior of Solutions of Second Order Nonlinear Delay Equations

John R. Graef* and Paul W. Spikes*

1. Introduction

Recently a number of papers giving sufficient conditions for oscillatory solutions of functional differential equations to converge to zero have appeared in the literature. Although this is closely related to the problem of finding sufficient conditions for all solutions to be nonoscillatory, considerably fewer results of this latter type are known. As examples of recent contributions on these problems we refer the reader to the papers [2–5 and 8–14] and the references contained therein.

Here we show that all (or certain classes) of the oscillatory solutions of the equation

\[(a(t)b(x'(t)))' + f(t, x(t), x(g(t))) = r(t, x(t)) \] (*)

tend to zero as \( t \to \infty \) under less restrictive conditions on the functions \( a, b, f \) and \( r \) than have been required by other authors. We also give sufficient conditions for (*) and

\[(a(t)x'(t))' + f(t, x(t), x(g(t))) = R(t) \] (**)

to be nonoscillatory without requiring the somewhat severe growth conditions on \( r(t, x) \) and \( R(t) \) usually needed by other authors. We have also included some examples illustrating our results and their relationship to previously known theorems.

2. Convergence to zero

Consider the equation

\[(a(t)b(x'(t)))' + f(t, x(t), x(g(t))) = r(t, x(t)) \] (1)

where \( a, g : [t_0, \infty) \to \mathbb{R}, b : \mathbb{R} \to \mathbb{R}, f : [t_0, \infty) \times \mathbb{R}^2 \to \mathbb{R} \) and \( r : [t_0, \infty) \times \mathbb{R} \to \mathbb{R} \) are continuous, \( a(t) > 0 \), \( g(t) \leq t \), and \( g(t) \to \infty \) as \( t \to \infty \). When \( b(y) \equiv y \) equation (1) will be denoted by

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\[ (a(t)x'(t))' + f(t, x(t), x(g(t))) = r(t, x(t)). \] (2)

We shall also utilize the conditions that there exist nonnegative constants \( p \leq 1, k_i \leq 1 \) and \( K_i \) for \( i = 1, 2, 3, 4 \), and functions \( q, h_1, h_2 \in C[t_0, \infty) \) such that

\[ |f(t, x, y)| \leq |q(t)| \left[ K_1 |x|^{k_1} |y|^{k_2} + K_2 |x|^{k_3} + K_3 |y|^{k_4} + K_4 \right] \] (3)

and

\[ |r(t, x)| \leq h_2(t) |x|^p + |h_1(t)| \] (4)

where \( k_1 + k_2 \leq 1 \). In addition, we shall ask that the function \( b \) satisfy \( b(0) = 0 \) and for \( y \neq 0 \)

\[ 0 < y/b(y) \leq A \] (5)

for some constant \( A \). The results given here pertain only to the continuuble solutions of the equations under consideration and we will use the same classification for such solutions as was used in [7]. That is, a solution \( x(t) \) will be called nonoscillatory if there exists \( t_1 \geq t_0 \) such that \( x(t) \neq 0 \) for \( t \geq t_1 \); the solution will be called oscillatory if for any given \( t_1 \geq t_0 \) there exist \( t_2 \) and \( t_3 \) with \( t_1 < t_2 < t_3 \), \( x(t_2) > 0 \) and \( x(t_3) < 0 \); and it will be called a Z-type solution if it has arbitrarily large zeros but is eventually nonnegative or nonpositive.

**Theorem 1.** If (3)-(5) hold and

\[ \int_{t_0}^{\infty} \left[ \int_s^{\infty} \left[ |q(u)| + h_2(u) + |h_1(u)| \right] du \right] ds < \infty, \] (6)

then every oscillatory or Z-type solution \( x(t) \) of (1) satisfies \( x(t) \to 0 \) as \( t \to \infty \).

**Proof.** Let \( x(t) \) be an oscillatory or Z-type solution of (1) on \([t_1, \infty)\), \( t_1 > \max \{t_0, 0\} \), and let \( t_2 \geq t_1 \) be such that \( g(t) \geq t_1 \) for \( t \geq t_2 \).

First suppose that \( x(t) \) is not bounded on \([t_1, \infty)\). By (6) there exists \( t_3 > t_2 \) so that \( x(t_3) = 0 \) and

\[ \int_{t_3}^{\infty} \left[ \int_s^{\infty} \left[ K |q(u)| + h_2(u) + |h_1(u)| \right] du \right] ds < 1/A \] (7)

where \( K = K_1 + K_2 + K_3 + K_4 \). Since \( x(t) \) is unbounded there exists \( T_1 > t_3 \) so that \( |x(T_1)| > 1 \) is the maximum value of \( x(t) \) for \( t_1 \leq t \leq T_1 \). Now \( x(t) \) is either oscillatory or Z-type so there is an interval \([c, d]\) containing \( T_1 \) with \( t_3 \leq c, x(c) = x(d) = 0 \), and \( x(t) \neq 0 \) on \((c, d)\). Choose \( T \) in \((c, d)\) satisfying \( |x(T)| = \max \{|x(t)| : c \leq t \leq d\} \geq |x(T_1)| \). Notice that since \( t_1 \leq g(t) \leq d \) for \( t_3 \leq t \leq d \), then both \( |x(t)| \) and \( |x(g(t))| \) are bounded from above by \( |x(T)| \) on \([t_3, d]\). Integrating (1) from \( t \) in \([c, T]\) to \( T \) yields

\[ a(t)b(x'(t)) = \int_c^T \left[ f(s, x(s), x(g(s))) - r(s, x(s)) \right] ds, \]
and multiplying the last equation by \( x'(t)/b(x'(t))a(t) \) and integrating we obtain

\[
x(T) = \int_c^T \left[ x'(s)/b(x'(s))a(s) \right] \int_s^T \left[ f(u, x(u), x(g(u))) - r(u, x(u)) \right] du ds.
\]

(8)

From (3)–(5) and the choice of \( T \), we have

\[
|x(T)| \leq A \int_c^T \left[ 1/a(s) \right] \int_s^T \left[ |K|q(u)| + h_2(u) |x(T)| + |h_1(u)| \right] du ds.
\]

But the last inequality implies that

\[
1 \leq A \int_c^T \left[ 1/a(s) \right] \int_s^T \left[ |K|q(u)| + h_2(u) + |h_1(u)| \right] du ds < 1.
\]

Therefore we conclude that \( x(t) \) is bounded, say \( |x(t)| < K_5 \) for \( t \geq t_1 \).

To complete the proof, assume that \( x(t) \) does not tend to zero as \( t \to \infty \), say \( \lim_{t \to \infty} |x(t)| = 2M > 0 \). Then (6), together with the fact that \( x(t) \) has arbitrarily large zeros, implies that there exists \( t_4 > t_2 \) such that

\[
\int_{t_4}^\infty \left[ 1/a(s) \right] \int_s^\infty \left[ K_6 |q(u)| + K_3 h_2(u) + |h_1(u)| \right] du ds < M/A
\]

holds where \( K_6 = K K_5 \). Since \( \limsup_{t \to \infty} |x(t)| = 2M \) there exists \( [c_1, d_1] \) such that \( c_1 > t_4, x(c_1) = x(d_1) = 0, x(t) \neq 0 \) on \( (c_1, d_1) \) and \( m = \max \{|x(t)| : c_1 \leq t \leq d_1 \} > M \). Choose \( z \) in \( (c_1, d_1) \) such that \( x(z) = m \). Then an argument similar to that used above leads to a contradiction, and the proof of the theorem is now complete.

It is easy to see that the equation

\[
[t x'(t)/(1 + \text{sech}(x'(t)))]' + (\sin t)(x(t)x(g(t)))^{1/3}/t^{3/2} = (\ln t)/t^2, \quad t \geq 4
\]

satisfies all the hypotheses of Theorem 1. Here we have \( k_1 = k_2 = 1/3, K_1 = 1, K_2 = K_3 = K_4 = 0, a(t) = t, q(t) = (\sin t)/t^{3/2}, h_2(t) = 0, h_1(t) = (\ln t)/t^2, b(y) = y/(1 + \text{sech} y), \) and \( A = 2 \).

REMARK. Theorem 1 includes as special cases Theorems 3.1 and 3.5 of Chiou [3], Theorems 4 and 5 of Kusano and Onose [8], Theorems 1–3 of Singh [10], Theorems 1 and 3 of Singh [11] and Corollary 3.1 of Singh [13] as well as the second order versions of Theorem 1 of Chen, Yeh, and Lin [2] and Theorem 3.2, Corollary 3.1 and Theorem 5.1 of Singh [13].

Our next result gives sufficient conditions for every unbounded oscillatory or \( Z \)-type solution of (1) to be slowly oscillating and for all moderately oscillating solutions of (1) to tend to zero as \( t \to \infty \). As defined in [9] and [11–13] we say that an oscillatory or \( Z \)-type solution \( x(t) \) is slowly oscillating if the set

\[
S_x = \{ t_\beta - t_\alpha : x(t_\beta) = x(t_\alpha) = 0, |x(t)| > 0 \quad \text{for} \quad t \in (t_\alpha, t_\beta) \}
\]

is unbounded and it is called moderately oscillating otherwise.
THEOREM 2. If in addition to (3)–(5)

\[ \int_{t_0}^{\infty} [\|q(s)\| + h_2(s) + |h_1(s)|] ds < \infty \]

(10)

and there exists a constant \( B_1 \) such that

\[ B_1 \leq a(t), \]

(11)

then every unbounded oscillatory or Z-type solution of (1) is slowly oscillating. Furthermore, every moderately oscillating solution \( x(t) \) of (1) satisfies \( x(t) \rightarrow 0 \) as \( t \rightarrow \infty \).

PROOF. Let \( x(t) \) be an unbounded oscillatory or Z-type solution of (1) on \([t_1, \infty)\), and suppose that \( x(t) \) is not slowly oscillating. Let \( N \) be an upper bound of the set \( S_x \); then from (10) we have the existence of a constant \( T_1 > t_1 \) such that

\[ \int_{T_1}^{\infty} [K|q(s)| + h_2(s) + |h_1(s)|] ds < B_1/AN \]

(12)

where \( K = K_1 + K_2 + K_3 + K_4 \). Now choose \( t_2 \geq T_1 \) so that \( x(t_2) = 0 \) and \( g(t) \geq T_1 \) for \( t \geq t_2 \). Since \( |x(t)| \) is unbounded, there exists \( T > t_2 \) so that \( |x(T)| = \max \{|x(t)| : T_1 \leq t \leq T\} > 1 \). We next choose an interval \((c_j, d_j)\) containing \( T \) and satisfying \( c_j \geq t_2 \), \( x(c_j) = x(d_j) = 0 \), and \( |x(t)| > 0 \) on \((c_j, d_j)\). Now let \( c \) in \((c_j, d_j)\) be such that \( M = |x(c)| = \max \{|x(t)| : c_j \leq t \leq d_j\} \). Then clearly \( |x(g(t))| \) is bounded above by \( M \) on \([c_j, d_j]\) and \( x(c) = \int_{c_j}^{d_j} x'(s) ds = -\int_{d_j}^{c_j} x'(s) ds \) so that

\[ 2M \leq \int_{c_j}^{d_j} |x'(s)| ds = \int_{c_j}^{d_j} |x'(s)|(a(s)/a(s))^{1/2} ds. \]

Therefore, from the Schwarz inequality, we have

\[
4M^2 \leq \left( \int_{c_j}^{d_j} \frac{1}{B_1} ds \right) \left( \int_{c_j}^{d_j} a(s) [x'(s)]^2 ds \right) \\
\leq (N/B_1) \int_{c_j}^{d_j} [(x'(s))^2 a(s) b(x'(s))/b(x'(s))] ds \\
\leq (AN/B_1) \int_{c_j}^{d_j} a(s) b(x'(s)) x'(s) ds.
\]

Upon integrating the last integral by parts we obtain

\[
4M^2 \leq (AN/B_1) \int_{c_j}^{d_j} x(s) [f(s, x(s), x(g(s))) - r(s, x(s))] ds \\
\leq (AN/B_1) \int_{c_j}^{d_j} M^2 [K|q(s)| + h_2(s) + |h_1(s)|] ds
\]

which implies that \( 4 \leq 1 \). This contradiction completes the proof that \( x(t) \) is slowly oscillating.
Next, let $x(t)$ be a moderately oscillating solution of (1). Then it follows from the argument above that $x(t)$ is bounded. It is not difficult to see that the boundedness of $x(t)$, together with (3)-(5) and (10)-(11), implies that $x'(t) \to 0$ as $t \to \infty$. But Philos and Staikos have shown [9; Lemma 3] that if $x(t)$ is moderately oscillating and $x'(t) \to 0$ as $t \to \infty$, then $x(t) \to 0$ as $t \to \infty$. This completes the proof of the theorem.

**Remark.** Sufficient conditions for oscillatory solutions which do not tend to zero as $t \to \infty$ to be slowly oscillating have been given by Chiou [3; Lemma 3.3] and Singh [11; Theorems 4 and 5] and [13; Theorem 4.1] for second order equations, and by Singh [12; Theorems 3.1 and 4.4] for $n$-th order equations. However, in addition to the hypotheses of Theorem 2 being less restrictive than those of Chiou and Singh, it is also applicable to equations not covered by their results.

The conclusion in Theorem 2 concerning the moderately oscillating solutions of (1) extends Theorem 6 in [8] and Theorem 4 in [10]. Also, Theorem 2 extends the 2-nd order versions of the theorem in [9] and Lemma 3.1 in [12] which give sufficient conditions for the bounded moderately oscillating solutions of $n$-th order equations to converge to zero as $t \to \infty$.

In the next two theorems we will take the special case of (1) when $b(y) = y$, i.e.

$$(a(t)x'(t))' + f(t, x(t), x(g(t))) = r(t, x(t)). \tag{2}$$

We will also utilize the linear equation

$$(a(t)u')' + q_1(t)u = 0 \tag{13}$$

with $q_1: [t_0, \infty) \to \mathbb{R}$ positive and continuous, and require that the inequality

$$x[q_1(t) - f(t, x, y)] \geq 0 \tag{14}$$

be satisfied in case (i) $xy \leq 0$ or (ii) $xy \geq 0$ and $|x| \geq |y|$.

**Theorem 3.** Suppose that (3), (4), (10), (11), and (14) hold. If, in addition, (13) is nonoscillatory,

$$xr(t, x) \geq 0 \quad \text{for all } x, \tag{15}$$

and there exists positive constants $L$ and $B_2$ such that

$$a(t) \leq B_2 \tag{16}$$

and

$$t - g(t) \leq L, \tag{17}$$

then every oscillatory solution $x(t)$ of (2) is bounded and satisfies $x'(t) \to 0$ and $[x(t) - x(g(t))] \to 0$ as $t \to \infty$. 

PROOF. Let \( x(t) \) be an oscillatory solution of (2) on \([t_1, \infty)\). Since (16) holds, (13) has a solution \( u(t) \) such that \( u(t)>0, u'(t)>0, \) and \((a(t)u'(t))'>0\) for \( t \in [t_2, \infty) \) for some \( t_2 \geq t_1 \). We may also assume that \( t_2 \) is large enough to ensure that \( g(t) \geq t_1 \) for \( t \geq t_2 \).

Now if \( x(t) \) is unbounded, then Theorem 2 implies that \( x(t) \) is slowly oscillating. Therefore there is an interval \([c, d]\) with \( t_2 \leq c, x(c)=x(d)=0, d-c>L \) and \( |x(t)| > 0 \) on \((c, d)\). If \( x(t)<0 \) on \((c, d)\), let \( M \) be the least number greater than or equal to \( d \) so that there is an interval \([M, M_1]\) with \( x(t)>0 \) on \((M, M_1)\). It then follows that \( x'(M) \geq 0 \) and that there exists \( N>M \) such that \( x'(N)=0 \) and \( x'(t)>0 \) on \((M, N)\). We also have from (17) that \( g(t) \geq c \) for \( t \geq M \). By multiplying (2) by \( u(t) \) and (13) by \( f(t, x(t), x(g(t)))/q_1(t) \) and subtracting, we obtain

\[
(a(t)x'(t))'u(t)-(a(t)u'(t))'f(t, x(t), x(g(t)))/q_1(t)=u(t)r(t, x(t))
\]

from which we have

\[
[a(t)x'(t)u(t)-a(t)u'(t)x(t)]' + (a(t)u'(t))'[x(t)q_1(t)-f(t, x(t), x(g(t)))]/q_1(t)
\]

\[
= u(t)r(t, x(t)).
\]

Notice that for each \( t \) in \([M, N]\) for which \( c \leq g(t) \leq M \) we have \( x(t)x(g(t)) \leq 0 \) and (14) implies that the second term in the preceding equation is nonpositive. Furthermore, if \( M \leq t \leq N \) and \( g(t) \geq M \), then \( 0 \leq x(g(t)) \leq x(t) \) by the choice of \( N \), and again (14) implies that the second term in the last equation is nonpositive. Hence an integration of this equation over \([M, N]\), together with (15), leads to the contradiction

\[
0>-a(N)u'(N)x(N)-a(M)x'(M)u(M)
\]

\[
\geq \int_M^N u(s)r(s, x(s))ds
\]

\[
\geq 0.
\]

Consequently we conclude that \( x(t) \) is bounded if \( x(t)<0 \) on \((c, d)\). The proof that \( x(t) \) is bounded if \( x(t)>0 \) on \((c, d)\) is similar and will be omitted.

To complete the proof of the theorem, notice first that (3), (4), and (10), together with the boundedness of \( x(t) \), imply \( \int_{t_1}^{\infty} |r(s, x(s))-f(s, x(s), x(g(s)))|ds < \infty \). Thus if \( \varepsilon \) is any positive number there exists \( T \geq t_2 \) so that \( x'(T)=0 \) and \( \int_T^{\infty} |r(s, x(s))-f(s, x(s), x(g(s)))|ds < B_1\varepsilon \). Then integrating (2) we obtain

\[
(a(t)x'(t))'=\int_T^t [r(s, x(s))-f(s, x(s), x(g(s)))]ds
\]

from which it follows by (11) and the choice of \( T \) that
Thus we conclude that $x'(t) \to 0$ as $t \to \infty$. Finally, observe that $x'(t) \to 0$ as $t \to \infty$, together with (15) and the mean-value theorem, implies $[x(t) - x(g(t))] \to 0$ as $t \to \infty$.

REMARK. It is interesting to observe that if (13) is non-oscillatory, (15) and (16) hold, and for all $(x, y)$ such that $xy \geq 0$, inequality (14) holds, then an argument similar to the one used in the proof of Theorem 3 shows that in fact (2) has no $Z$-type solutions.

THEOREM 4. Let the hypotheses of Theorem 3 be satisfied. Suppose also that there exist continuous functions $q_3 : [t_0, \infty) \to \mathbb{R}$ and $\delta : [t_0, \infty) \times \mathbb{R}^2 \to \mathbb{R}$ such that $q_3$ is either non-negative or non-positive, and for all positive constants $\gamma, \varepsilon$, and $T_0 \geq t_0$ there exists $\lambda > 0$ such that if $|x| + |y| \leq \gamma$ and $|x - y| < \lambda$, then

$$|\delta(t, x, y)| < \varepsilon$$

and

$$f(t, x, y) - r(t, x) = q_3(t)[x - \delta(t, x, y)]$$

for all $t \to T_0$. If, in addition, the linear equation

$$(a(t)z')' + q_3(t)z = 0$$

is non-oscillatory, then every oscillatory solution $x(t)$ of (2) satisfies $x(t) \to 0$ as $t \to \infty$.

PROOF. Let $x(t)$ be an oscillatory solution of (2) on $[t_1, \infty)$; then by Theorem 3 $x(t)$ is bounded say $|x(t)| < \gamma/2$ and $|x(g(t))| < \gamma/2$ on $[t_1, \infty)$ for some constant $\gamma > 0$. Now let $m > 0$ be given and choose $0 < \varepsilon < m$; then there exists $\lambda > 0$ so that (18) holds for $t \geq t_1$. From Theorem 3, there exists $T \geq t_1$ so that $|x(t) - x(g(t))| < \lambda$ for $t \geq T$.

Next, observe that $x(t)$ is a solution of the linear equation

$$(a(t)x')' + q_3(t)x = q_3(t)\delta(t, x(t), x(g(t)))$$

on $[T, \infty)$, and hence $w(t) = x(t) - m$ is a solution of the equation

$$(a(t)z')' + q_3(t)z = q_3(t)[\delta(t, x(t), x(g(t))) - m].$$

(21)

Since the right member of (21) does not change sign on $[T, \infty)$ and (20) is non-oscillatory, then (22) is also nonoscillatory (see [6]). Thus $w(t)$ is nonoscillatory. If $q_3(t) \geq 0$ then the right member of (21) is nonpositive for $t \geq T$. Furthermore, if $w(t)$ is eventually positive, say $w(t) > 0$ for $t \geq T_1 \geq T$, then $(a(t)w'(t))' = (a(t)x'(t))'$ \leq 0 for $t \geq T_1$. But this is impossible in view of the fact that $x(t)$ is oscillatory.
Thus \( w(t) = x(t) - m < 0 \) for all sufficiently large \( t \). Since \( m \) is an arbitrary positive number, this implies that \( \lim_{t \to \infty} x(t) = 0 \). By taking \( m < 0 \) and \( 0 < \epsilon < |m| \) we can show, in a similar fashion, that eventually \( w(t) = x(t) - m > 0 \) and that \( \lim_{t \to \infty} x(t) = 0 \). Therefore we conclude that \( x(t) \to 0 \) as \( t \to \infty \) in case \( q_3(t) \geq 0 \). The proof for the case \( q_3(t) \leq 0 \) is similar and will be omitted.

**Remark.** Other authors (see for example [2, 3 and 8–14]) have obtained sufficient conditions for oscillatory solutions to converge to zero under integral conditions more restrictive than (10). For example Theorem 4 applies to the equation

\[
x''(t) + x(t) \tanh \left[ x(t) - x(t-L) \right]/5t^2 = x(t)/10t^{3/2}, \ t > 1
\]

where \( L \) is a positive constant. Here \( q_1(t) = 1/5t^2, q_3(t) = -1/10t^{3/2} \), and \( \delta(t, x, y) = 2x \tanh (x-y)/t^{1/2} \). Notice that \( \int_{t_0}^{\infty} |q_1(s)|ds < \infty \) but \( \int_{t_0}^{\infty} s|q_1(s)|ds = \infty \) so none of the results in [2, 3 or 8–14] apply to this equation.

Of particular interest is the observation that any linear delay equation

\[
(a(t)x'(t))' + q(t)x(g(t)) = 0
\]

with \( 0 < B_1 < a(t) < B_2, 0 \leq t-g(t) < L, \) and \( 0 \leq q(t) \leq q_1(t) \), where \( B_1, B_2 \) and \( L \) are positive constants and such that the equation \( (a(t)u')' + q_1(t)u = 0 \) is nonoscillatory satisfies the hypotheses of Theorem 4.

3. Nonoscillation

We now turn to the problem of obtaining nonoscillation results for (1) and some of its special cases.

**Theorem 5.** If, in addition to (5) and (15), there exist continuous functions \( Q \) and \( F \) such that

\[
|f(t, x, y)| \leq Q(t)|xF(x, y)|
\]

and

\[
\int_{t_0}^{\infty} [1/a(s)] \int_{s}^{\infty} Q(u)duds < \infty,
\]

then all bounded solutions of (1) are nonoscillatory.

**Proof.** Assume that (1) has a bounded oscillatory or Z-type solution on \([t_1, \infty)\). Then (22) implies there exists a constant \( N > 0 \) so that

\[
|f(t, x(t), x(g(t)))| \leq |x(t)|Q(t)|F(x(t), x(g(t)))| \leq NQ(t)|x(t)|,
\]

for \( t \geq t_1 \) and (23) implies there exists \( t_2 \geq t_1 \) such that \( \int_{t_2}^{\infty} [1/a(s)] \int_{s}^{\infty} Q(u)duds < \)
1/2NA. Furthermore, since $x(t)$ is oscillatory or $Z$-type, there exist $t_3$ and $t_4$ satisfying $t_2 < t_3 < t_4$, $x(t_3) = x(t_4) = 0$, and $|x(t)| > 0$ on $(t_3, t_4)$. Let $d$ in $(t_3, t_4)$ be chosen so that $|x(d)| = \max \{|x(t)|: t_3 \leq t \leq t_4\}$. Integrating (1) twice we obtain equation (8) with $c$ and $T$ replaced by $t_3$ and $d$ respectively. An application of (5) and (15) yields

$$|x(d)| \leq A \int_{t_3}^{d} [1/a(s)] \int_{s}^{d} NQ(u)|x(u)|\,du\,ds.$$  
This implies that

$$1 \leq AN \int_{t_3}^{d} [1/a(s)] \int_{s}^{d} Q(u)\,du\,ds < 1/2$$  
and this contradiction completes the proof of the theorem.

**Theorem 6.** If (5), (15), (22), and (23) hold and the function $F$ in (22) is bounded in $\mathbb{R}^2$, the (1) is nonoscillatory.

The details of the proof of Theorem 6 are essentially the same as those of Theorem 5 and will be omitted.

Theorem 5 can be applied to the example

$$(x'(t)/t)' + (\ln t)x^3(t)x^5(g(t))/t^4 = t^2 \tanh (x(t)), \quad t > 1$$

while the equation

$$(x'(t)/t)' + (\ln t)x^3(t)x(g(t))/t^4(1 + x^2(t))(1 + x^2(g(t))) = t^2 \tanh (x(t)), \quad t > 1$$

satisfies the hypotheses of Theorem 6.

**Remark.** In Theorems 5 and 6 above no growth conditions are imposed on the size of $r(t, x)$. This differs significantly from nonoscillation results for delay equations obtained by other authors (see for example [4] and [5]). It should also be pointed out that there have been several erroneous attempts to find nonoscillation criteria for higher order equations. (In particular Theorem 2 in [1] is incorrect.) In this regard see the discussion in [4].

Next we obtain some nonoscillation results for equation (2) when $r(t, x) = R(t)$, i.e. for the equation

$$(a(t)x'(t))'+f(t, x(t), x(g(t))) = R(t). \quad (24)$$

The following theorem is similar to Theorem 1 in [6].

**Theorem 7.** Let equation (13) be nonoscillatory and condition (14) hold for all $(x, y)$ in $\mathbb{R}^2$. If $R(t) \geq 0$ then all solutions of (24) are either nonoscillatory or nonpositive (nonnegative) $Z$-type. If in addition

$$xf(t, x, y) \geq 0 \quad \text{for } xy \geq 0,$$

then (24) is nonoscillatory.
PROOF. Let \( R(t) \geq 0 \) and assume that (24) has a solution \( x(t) \) on \([t_1, \infty)\) that is either oscillatory or nonnegative Z-type. Let \( u(t) \) be a solution of (13) satisfying \( u(t) > 0 \) and \( u'(t) > 0 \) for \( t \geq t_2 \). Then there are consecutive zeros \( t_3 \) and \( t_4 \) of \( x(t) \) so that \( t_2 < t_3 < t_4 \), \( x'(t_3) = 0 \) and \( x(t_4) > 0 \) for \( t \) in \((t_3, t_4)\). Let \( c \) in \((t_3, t_4)\) be such that \( x(c) = \max \{x(t) : t_3 \leq t \leq t_4\} \), then \( x'(c) = 0 \). Multiplying (13) by \( f(t, x(t), x(g(t))) / q_1(t) \) and (24) by \( u(t) \) and subtracting, we have

\[
(a(t) [x'(t)u(t) - u'(t)x(t)])' + (a(t)u'(t))[x(t)q_1(t) - f(t, x(t), x(g(t)))] / q_1(t) = u(t)R(t).
\]

Notice that \( (a(t)u'(t))' < 0 \), which, together with (14), implies that the second term in the left member of the last equation is nonpositive. Integrating this equation over \([t_3, c]\) yields the contradiction

\[
0 > -a(c)u'(c)x(c) - a(t_3)x'(t_3)u(t_3)
\]

\[
\geq \int_{t_3}^{c} u(s)R(s)ds.
\]

Thus we conclude that all solutions of (24) are either nonoscillatory or nonpositive Z-type in case \( R(t) \geq 0 \). Now if \( R(t) \geq 0 \), (25) holds, and \( x(t) \) is a nonpositive Z-type solution of (24), then there exists \( T \geq t_1 \) such that \( x(t) \leq 0 \) and \( x(g(t)) \leq 0 \) for \( t \geq T \). Then \( (a(i)x'(i))' \geq 0 \) for \( t \geq T \) which is impossible for a solution of this type. This completes the proof of the theorem for the case \( R(t) \geq 0 \).

The proof for the case \( R(t) \leq 0 \) is similar and will be omitted.

If \( q_1(t) > Q(t) > 0 \) and (13) is nonoscillatory, then Theorem 7 implies that every solution \( x(t) \) of

\[
(a(t)x'(t))' + Q(t)x(t)(\sin x(t)) \tanh (x(g(t))) = R(t)
\]

is either nonoscillatory or nonpositive Z-type if \( R(t) \geq 0 \) and is either nonoscillatory or nonnegative Z-type in case \( R(t) \leq 0 \), and that the equation

\[
(a(t)x'(t))' + Q(t)x(t)(\frac{1}{\cosh x(t)})^2 = R(t)
\]

is nonoscillatory if \( R(t) \) is eventually either nonnegative or nonpositive.

Once again no restriction on the size of \( R(t) \) or relationship between \( R(t) \) and \( f(t, x, y) \) is needed in Theorem 7. In the next theorem \( R(t) \) is required to be small.

**Theorem 8.** Let (13) be nonoscillatory, (3) hold, (6) hold with \( h_2(t) \equiv 0 \) and \( h_1(t) \equiv R(t) \), and there exist a constant \( m > 0 \) such that (14) holds whenever \( |x| + |y| < m \). If \( R(t) \geq 0(R(t) \leq 0) \), then every solution of (24) is either nonoscillatory or nonpositive Z-type (nonoscillatory or nonnegative Z-type). If in addition (25) holds, then (24) is nonoscillatory.
PROOF. Notice that all the hypotheses of Theorem 1 are satisfied. Thus if \( x(t) \) is an oscillatory or Z-type solution of (24), then Theorem 1 implies that \( x(t) \to 0 \) as \( t \to \infty \). Proceeding as in the proof of Theorem 7, we again obtain that equation (24) is nonoscillatory.

References


