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Continuous Linear Functional on Closed 
Two-sided Ideals of C*-algebras

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G. K. Pedersen [2] have proved that any positive linear functional on any order-related C*-subalgebra of a C*-algebra A has a unique norm-preserving extension to A. In [3], we showed the converse of this result and hence obtained a characterization of order-related C*-subalgebras. We also showed that any continuous linear functional on every order-related C*-subalgebra of A has a unique norm-preserving extension to A.

In this note, we state a characterization of closed two-sided ideals I of A (opposed to order-related C*-subalgebras) in terms of norm-preserving extension of special kind of linear functionals on I. As an application of this characterization, we further show that a C*-algebra A is dual if and only if the Dixmier's decomposition of any element of the third dual space of $A^{***}$ of A is orthogonal.

Throughout this note, let A be a C*-algebra and M its enveloping von Neumann algebra. We consider A as lying in M under the canonical embedding. If B is a C*-subalgebra of A, then any increasing approximate identity of B converges ultraweakly to a unique projection $E_B$ in M, and $E_B$ said to be the support of B. Set $A(E_B) = \{x \in A : xE_B = E_Bx = x\}$. Then a C*-subalgebra B of A is order-related if and only if $B = A(E_B)$. Also $E_B$ is central provided B is a two-sided ideal of A.

Now our main result is the following

**Theorem 1.** In order that a C*-subalgebra B of A to be a two-sided ideal of A it is necessary and sufficient that $\|f + g\| = \|f\| + \|g\|$ for any continuous linear functionals f and g on A with $\|f|B\| = \|f\|$ and $g|B = 0$.

**Proof.** Let $E_B$ be the support of B in M. Suppose first that B is a two-sided ideal of A and let $f$, $g$ be continuous linear functionals on A such that $\|f|B\| = \|f\|$ and $g|B = 0$. Set $h = f + g$. Since $E_B$ is central, $\|h\| = \|E_B \cdot h\| + \|(1 - E_B) \cdot h\|$ by means of the enveloping polar decomposition of $h$ (cf. [1, 12, 2.7]). Note that $E_B \cdot h$ and $f$ are norm-preserving extensions of $f|B$. Then $E_B \cdot h = f$ and hence $(1 - E_B) \cdot h = g$ from [3, Theorem 2]. Thus the necessity is proved.

Assume next that $\|f + g\| = \|f\| + \|g\|$ for any continuous linear functionals

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We claim that $B$ is order-related. In fact, let $\phi$ be any continuous linear functional on $B$ and $f, g$ be norm-preserving extensions of $\phi$ to $A$. Set $h = f - g$. Then $h|B = 0$ and $\|f|B\| = \|g\|$. By the assumption, $\|f\| = \|g\| + \|h\|$ and so $h = 0$. In other word, $\phi$ has a unique norm-preserving extension to $A$. Therefore $B$ is order-related from [3, Theorem 1].

Thus, to show that $B$ is a two-sided ideal of $A$, we only show $E_B$ is central. Actually, let $f$ be any positive linear functional on $A$ and set

$$f_n = E_B \cdot f \cdot E_B,$$

$$f_s = (1 - E_B) \cdot f \cdot E_B + E_B \cdot f \cdot (1 - E_B) + (1 - E_B) \cdot f \cdot (1 - E_B).$$

Then $f_n$ is positive and so $E_{f_n} \leq E_B$. Here $E_{f_n}$ denotes the support of $f_n$ in $M$. Obviously $f = f_n + f_s$ and $f_s|B = 0$. We also have, from [1, 2.1.5],

$$\|f_n\| = \|f(E_B)\| = \lim_{\lambda} f(b_\lambda) = \lim_{\lambda} f_n(b_\lambda) = \|f_n|B\|,$$

where $\{b_\lambda\}$ is an increasing approximate identity of $B$. By the assumption, $\|f\| = \|f_n\| + \|f_s\|$. This equality implies that $f_s$ is also positive. Therefore $E_{f_s} \leq 1 - E_B$ because $f_s(E_B) = 0$. We thus obtain that

$$g(xE_B) = g_n(xE_B) + g_s(xE_B) = g_n(x)$$

$$= g_n(E_Bx) + g_s(E_Bx)$$

$$= g(E_Bx)$$

for all positive linear functional $g$ on $A$ and all $x \in A$. In other word, $E_B$ is central and the sufficiency is proved.

**REMARK.** As be seen in the above proof, if $B$ is a closed two-sided ideal of $A$, then for any continuous linear functional $h$ on $A$, there exist unique continuous linear functionals $f$ and $g$ on $A$ such that $h = f + g$, $\|f|B\| = \|f\|$, $g|B = 0$ and $\|h\| = \|f\| + \|g\|$.  

Let $R$ be a $C^*$-subalgebra of $M$ containing $A$. For each continuous linear functional $f$ on $R$, we denote by $N_R(f)$ a unique ultraweakly continuous extension of $f|A$ to $R$ and let $S_R(f) = f - N_R(f)$. We then have the following

**COROLLARY 2.** In order that $A$ to be a two-sided ideal of $R$, it is necessary that $\|f\| = \|N_R(f)\| + \|S_R(f)\|$ for any continuous linear functional $f$ on $R$ and sufficient that $\|f\| = \|N_R(f)\| + \|S_R(f)\|$ for any positive linear functional $f$ on $R$.

**PROOF.** Let $f$ be a continuous linear functional on $R$. Observe that $\|N_R(f)\| = \|f|A\| = \|N_R(f)|A\|$ and $S_R(f)|A = 0$. Then from Theorem 1, $\|f\| = \|N_R(f)\| + \|S_R(f)\|$ whenever $A$ is a two-sided ideal of $R$.

Suppose next that $\|f\| = \|N_R(f)\| + \|S_R(f)\|$ for positive linear functional $f$
on $R$. By [3, Corollary 5], $A$ is an order-related $C^*$-subalgebra of $R$. Then we have only to show that $E_A$ is central in the enveloping von Neumann algebra of $R$. Let $f$ be any positive linear functional on $R$. Then $N_R(f)$ is also positive and

$$N_R(f)(E_A) = \lim N_R(f)(a_\lambda) = \lim f(a_\lambda) = \|f\| A = \|N_R(f)\|, \text{ where } \{a_\lambda\} \text{ is an increasing approximate identity of } A.$$

Then $E_{N_R(f)} \leq E_A$. Since $N_R(f)$ is positive and $f = \|N_R(f)\| + S_R(f)$, $S_R(f)$ is also positive. Now $E_{S_R(f)} \leq 1 - E_A$ because $S_R(f)|A = 0$. Therefore, after the manner of the proof of Theorem 1, we obtain that $E_A$ is central and the proof is complete.

We now consider the dual space $A^*$ of $A$ as lying in the third dual space $A^{***}$ of $A$ under the canonical embedding. Let $A^c = \{F \in A^{***} : F(x) = 0 \text{ if } x \in A\}$.

**Corollary 3.** The following statements are equivalent:

(a) $A$ is a dual algebra.

(b) $A^{***}$ is isometrically isomorphic to $A^* \oplus A^c$.

**Proof.** $A^{***}$ is always isomorphic to $A^* \oplus A^c$ as vector spaces from the Dixmier’s theorem. Identifying $A^{**}$ and $M$, the corollary follows immediately from [4, Theorem 5.1] and Corollary 2.

**References**


