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ROSEリポジトリいばらき （茨城大学学術情報リポジトリ）
A Remark on Archbold's Result, II

Sin-ei TAKAHASI*

1. R. J. Archbold [3] has established that in order for an arbitrary C*-algebra to be quasi-central it is necessary and sufficient that A has an approximate identity each element of which belongs to the center of A. In, furthermore, [6] we have given a generalization of the above Archbold's result.

In this note, our purpose is to give an extension of the above result given in [6]. Now let A be an arbitrary C*-algebra and A** the second dual space of A. We may regard A** as a von Neumann algebra (W*-algebra) containing A as C*-subalgebra. If B is a C*-subalgebra of A, then any increasing approximate identity of B converges ultraweakly to a unique projection EB in A**, and EB is said to be the support of B. The result given in [6] is the following

THEOREM A. Let A be a C*-algebra and B a C*-subalgebra of A such that EB is central. Then the following conditions are equivalent:

(i) No primitive ideal of A contains B,

(ii) A has an approximate identity each element of which belongs to B.

If the condition (ii) holds, then EB is necessarily central. Actually, it is equal to the identity element of A**. However, the condition (i) does not necessarily imply that EB is central. In fact, let A be the algebra of all compact linear operators on an infinite dimensional Hilbert space and therefore A** coincides with the algebra of all bounded linear operators on the Hilbert space. Let p be a non-zero finite dimensional projection in A. Set B=pAp and hence B is a non-zero C*-subalgebra of A. Notice that the zero ideal is the only primitive ideal of A and therefore the condition (i) holds. However, the support EB of B is equal to p and p is not evidently central.

In order to take off these misfortunes, we will introduce a central support of a projection in A**. For an arbitrary projection p in A**, denote by Z[p] the central support of p, that is the smallest central projection which majorizes p in A**.

2. THEOREM. Let A be a C*-algebra and B a C*-subalgebra of A. Let Δ be a subset of P(A), the set of all pure states of A. If B is not contained in Ker πf for each f ∈ Δ, then EΔ ≤ Z[EB]. Conversely, if EΔ ≤ EB, then B is not

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contained in Ker \( \pi_f \) for each \( f \in \Delta \).

Here \( \pi_f \) for \( f \in P(A) \) denotes the canonical cyclic representation of \( A \) on a Hilbert space \( H_f \) associated with \( f \) and \( E_f = \vee \{ E_f : f \in \Delta \} \), where \( E_f \) is the support of \( f \) in \( A^{**} \).

**Proof.** By Proposition 12.1.5 in [5], any representation \( \pi \) of \( A \) has a unique ultraweakly continuous extension \( \tilde{\pi} \) to \( \text{cl}(A) \) such that \( \tilde{\pi}(\text{cl}(A)) = \text{cl}(\pi(A)) \), where \( \text{cl} \) denotes the closure in the relevant weak operator topology. Now suppose first that \( B \) is not contained in Ker \( \pi_f \) for each \( f \in \Delta \). Let \( f \) be an arbitrary element of \( \Delta \). Then \( \tilde{\pi}_f(E_B) \neq 0 \) and hence \( \tilde{\pi}_f(Z[E_B]) \neq 0 \). Since \( \pi_f(A) \) is a strictly irreducible \( C^* \)-algebra on \( H_f \) and \( \tilde{\pi}_f(Z[E_B]) \) is a non-zero projection which commutes each element of \( \pi_f(A) \), it follows that \( \tilde{\pi}_f(Z[E_B]) \) is the identity operator on \( H_f \) and hence we have

\[
f(z[E_B]) = (\tilde{\pi}_f(Z[E_B])z_f|z_f) = (z_f|z_f) = 1,
\]

where \( z_f \) is the canonical cyclic vector associate with \( f \) in \( H_f \). Hence \( E_f \leq Z[E_B] \).

We thus obtain that \( E_d \leq Z[E_B] \) since \( f \) is arbitrary.

Suppose conversely that \( E_d \leq E_B \). If there exists an element \( f_0 \) of \( \Delta \) such that \( B \) is contained in Ker \( \pi_{f_0} \), then we have

\[
f_0(b) = (\pi_{f_0}(b)z_{f_0}|z_{f_0}) = 0
\]

for all \( b \in B \). In the other words, \( E_{f_0}E_B = 0 \). It follows that \( E_{f_0} = E_{f_0}E_B E_{f_0} \leq E_{f_0}E_B E_{f_0} = 0 \) and so \( E_{f_0} = 0 \). This contradiction completes the proof.

3. **Remark.** In the above theorem, we assume that \( \Delta = P(A) \) and denote by \( z \) the central projection in \( A^{**} \) which is the supremum of all the minimal projections in \( A^{**} \) [cf. 2, p. 278 or 4, p. 126]. We then have \( E_d = z \). In addition, assume that \( E_B \) is central. If no primitive ideal of \( A \) contains \( B \), then \( z \leq E_B \) from our theorem. Choose an approximate identity \( \{ b_\lambda \} \) of \( B \) arbitrarily. For each element \( f \) of \( P(A) \), we have

\[
\lim_{\lambda} f(b_\lambda) = f(E_B) = f(z) = 1.
\]

Therefore, [1, Theorem 5] applies to give that \( \{ b_\lambda \} \) is an approximate identity of \( A \). If, conversely, \( \{ c_\lambda \} \) is an approximate identity of \( A \), lying \( B \), then \( E_B \) is the identity element of \( A^{**} \) and hence \( E_{P(A)} \leq E_B \). It follows from our theorem that no primitive ideal of \( A \) contains \( B \). Thus we see that our theorem implies immediately Theorem A.
References