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One-Parameter Semigroups of the Gaussian and Poisson Integral Transforms on the $W^2$-space

Katsuo Takano*

Introduction. Let $W^2$ be the totality of Lebesgue measurable functions such that

$$\int_{-\infty}^{\infty} \frac{|f(x)|^2}{x^2 + 1} \, dx < \infty.$$  

When $f$ and $g$ are in $W^2$, we define the inner product of $f$ and $g$ by

$$(f, g) = \int_{-\infty}^{\infty} f(x)\overline{g(x)} \frac{1}{x^2 + 1} \, dx.$$  

Introducing the operations of addition and scalar multiplication in $W^2$ as usual, we can show that the space $W^2$ is the Hilbert space with norm $\|f\| = [(f, f)]^{1/2}$. Let $t > 0$ and let

$$(T(t)f)(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{(x-y)^2}{4t} \right) f(y) \, dy,$$

$T(0)f = f$

and

$$(P(t)f)(x) = \int_{-\infty}^{\infty} \frac{t}{\pi} \frac{1}{(x-y)^2 + t} f(y) \, dy,$$

$P(0)f = f$

for $f$ in $W^2$. If we choose the $L^2(R)$-space instead of $W^2$, we know that the family $\{T(t): 0 \leq t < \infty\}$ is a semigroup of class $(C_0)$ on $L^2(R)$ and also the family $\{P(t): 0 \leq t < \infty\}$ is a semigroup of class $(C_0)$ on $L^2(R)$, and that the infinitesimal generator of $\{T(t): 0 \leq t < \infty\}$ is the differential operator $\frac{d^2}{dx^2}$ and the infinitesimal generator of $\{P(t): 0 \leq t < \infty\}$ is the composition of the differential operator $\frac{d}{dx}$ and the Hilbert transform $C$, that is, $\frac{d}{dx} \cdot C$, and that

$$\frac{d^2}{dx^2} = -\left[ \frac{d}{dx} \cdot C \right]^2$$

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holds for \( f \) in the domain \( \left( \frac{d}{dx} \right)^2 \). (See [3]) Hence it is the purpose of this short note to show that similar relation also holds between the families of linear operators \( \{ T(t) : 0 \leq t < \infty \} \) and \( \{ P(t) : 0 \leq t < \infty \} \) on \( W^2 \).

§ 1. The semigroup \( \{ T(t) : 0 \leq t < \infty \} \) associated with the Gaussian kernel

In this section let us show that the family \( \{ T(t) : 0 \leq t < \infty \} \) of linear operators on \( W^2 \) is a semigroup of class \( (C_0) \).

**Lemma 1.** \( T(t) \) is a linear bounded operator on \( W^2 \) to itself and

\[
\omega_0 = \lim_{t \to \infty} \frac{1}{t} \log \| T(t) \| = 0.
\]

**Proof.** Let \( f \) be in \( W^2 \). By the Schwartz inequality we have

\[
|(T(t)f)(x)|^2 \leq (4\pi t)^{-1/2} \int_{-\infty}^{\infty} \exp \left( -\frac{(x-y)^2}{4t} \right) |f(y)|^2 dy
\]

\[
\cdot (4\pi t)^{-1/2} \int_{-\infty}^{\infty} \exp \left( -\frac{(x-y)^2}{4t} \right) dy
\]

\[
= (4\pi t)^{-1/2} \int_{-\infty}^{\infty} \exp \left( -\frac{(x-y)^2}{4t} \right) |f(y)|^2 dy.
\]

Hence by the Fubini theorem we obtain

\[
\int_{-\infty}^{\infty} \frac{|(T(t)f)(x)|^2}{x^2+1} dx \leq \int_{-\infty}^{\infty} |f(y)|^2 \left( \int_{-\infty}^{\infty} (4\pi t)^{-1/2} \exp \left( -\frac{(x-y)^2}{4t} \right) \frac{dx}{x^2+1} \right) dy
\]

\[
\leq \| f \|^2 \sup_{y \in \mathbb{R}} \left\{ (y^2+1) \int_{-\infty}^{\infty} (4\pi t)^{-1/2} \exp \left( -\frac{(x-y)^2}{4t} \right) \frac{dx}{x^2+1} \right\}.
\]

(1.1)

Let us prove that

\[
\sup_{y \in \mathbb{R}} \left\{ (1+y^2) \int_{-\infty}^{\infty} (4\pi t)^{-1/2} \exp \left( -\frac{(x-y)^2}{4t} \right) \frac{dx}{x^2+1} \right\}
\]

(1.2)

is a finite value. By the Parseval theorem we see that

\[
\int_{-\infty}^{\infty} (4\pi t)^{-1/2} \exp \left( -\frac{(x-y)^2}{4t} \right) \frac{dx}{x^2+1}
\]

\[
= \frac{1}{\pi} \int_{-\infty}^{\infty} \exp (2\pi i xy - 4\pi^2 tx^2 - 2\pi |x|) dx
\]

\[
= \frac{2}{\pi} \int_{0}^{\infty} \cos (2\pi yx) \exp (-4\pi^2 tx^2 - 2\pi x) dx
\]

(1.3)

To prove that (1.2) is a finite value, by (1.3) it suffices to prove that
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\[
\sup_{y \in R} \left\{ 2\pi y^2 \int_0^\infty \cos (2\pi y x) \exp (-4\pi^2 t x^2 - 2\pi x) dx \right\} \quad (1.4)
\]

is a finite value. We see that

\[
2\pi \int_0^\infty \cos (2\pi x) \exp (-4\pi^2 t x^2 - 2\pi x) dx 
\leq 1 + 2\pi \int_0^\infty (1 + 8\pi t + 16\pi^2 t^2 x^2 + 2t)e^{-2\pi x} dx 
\]

\[= 2 + 6t + 8t^2. \quad (1.5)\]

From the above arguments and (1.1) it follows that

\[
\int_{-\infty}^\infty \frac{|(T(t)f)(x)|^2}{x^2 + 1} dx \leq \|f\|^2 (3 + 6t + 8t^2).
\]

Hence we see that \(\|T(t)\|\) is uniformly bounded in \(t\) at the neighborhood of \(t=0\) and

\[
\omega_0 = \lim_{t \to 0} \frac{1}{t} \log \|T(t)\| = 0.
\]

Q.E.D.

Let us denote the differentiation \(d/dx\) by \(D\).

**THEOREM 1.** The family \(\{T(t): 0 \leq t < \infty\}\) of linear operators on \(W^2\) to itself is a semigroup of class \((C_0)\) and its infinitesimal generator \(A\) with domain \(D(A)\) is given by the following form;

\[
D(A) = \{f \in W^2: f(x), f'(x) \text{ are absolutely continuous and } f'(x), f''(x) \in W^2\}
\]

and

\[
Af = D^2 f
\]

for \(f\) in \(D(A)\).

**PROOF.** 1. It is easy to prove that the semigroup property and the strong continuity of \(\{T(t): 0 \leq t < \infty\}\).

2. Infinitesimal generator \(A\) and its domain \(D(A)\): From Lemma 1 and [2], the resolvent \(R(\lambda; A)\) of the infinitesimal generator \(A\) is given by

\[
R(\lambda; A)f = \int_0^\infty e^{-\lambda t} T(t) f dt, (f \in W^2)
\]

(1.6)

for \(\lambda > \omega_0 = 0\). For simplicity let \(\lambda = 1\). Then

\[
D(A) = \{g = R(1; A)f: f \in W^2\}
\]

Take a function \(f\) in \(W^2\) and let \(g = R(1; A)f\). Let us show that \(g(x), g'(x)\) are absolutely continuous and \(g', g'' \in W^2\). From (1.6) and by the Fubini theorem we obtain that
\[ g(x) = \frac{1}{2} \int_{-\infty}^{\infty} f(y)e^{-|x-y|}dy \]

\[ = \frac{1}{2} \int_{-\infty}^{\infty} f(y)e^{-y}dy e^{x} + \frac{1}{2} \int_{-\infty}^{x} f(y)e^{y}dy e^{-x}. \quad (1.7) \]

Hence

\[ g'(x) = \frac{1}{2} \int_{-\infty}^{\infty} f(y)e^{-y}dy e^{x} - \frac{1}{2} \int_{-\infty}^{x} f(y)e^{y}dy e^{-x} \quad (1.8) \]

and

\[ g''(x) = -f(x) + g(x) \quad (1.9) \]

for almost all \( x \in \mathbb{R} \).

From (1.7), (1.8) it is seen that \( g(x), g'(x) \) are absolutely continuous.

Next let us prove that \( g'(x) \) is in \( W^2 \). It holds that

\[ |g'(x)| \leq \frac{1}{2} \int_{-\infty}^{\infty} |f(y)|e^{-|x-y|}dy. \]

By the Schwartz inequality and the Fubini theorem we see that

\[ \int_{-\infty}^{\infty} \frac{|g'(x)|^2}{x^2 + 1} dx \leq \frac{1}{4} \int_{-\infty}^{\infty} \frac{1}{1 + x^2} \left[ \int_{-\infty}^{\infty} \frac{|f(y)|}{1 + (x-y)^2} \left( 1 + (x-y)^2 \right)^{1/2} e^{-|x-y|}dy \right]^2 dx \]

\[ \leq \frac{1}{4} \int_{0}^{\infty} (1 + x^2)e^{-2x} dx \int_{-\infty}^{\infty} |f(y)|^2 \left\{ \int_{-\infty}^{\infty} \frac{1}{1 + x^2} \frac{1}{1 + (x-y)^2} dx \right\} dy \]

\[ = \frac{3}{8} \int_{-\infty}^{\infty} \frac{|f(y)|^2}{y^2 + 4} dy < \infty. \]

From this fact we also see that \( g(x) \) is in \( W^2 \) and \( g''(x) \) is in \( W^2 \). Hence we find from (1.9) and from the same manner as in [7. Pages 243–244] that

\[ D(A) = \{ f \in W^2 : f \text{ and } f' \text{ are absolutely continuous and } f', f'' \in W^2 \} \]

and \( Af = f'' \) for \( f \) in \( D(A) \).

Q. E. D.

§ 2. The semigroup \( \{ P(t) : 0 \leq t < \infty \} \) associated with the Poisson kernel

For simplicity let

\[ n(x) = \frac{f(x)}{x - i}, \quad g(x) = \frac{f(x)}{(u - i)^2} \]

for \( f \) in \( W^2 \). Let \( Cn \) denote the Hilbert transform of \( n \), that is,

\[ (Cn)(x) = \lim_{\epsilon \to +0} \frac{1}{\pi} \int_{-\epsilon}^{\epsilon} \frac{n(x + u) - n(x - u)}{u} du. \]
Let $F_n$ denote the Fourier transform of $n$, that is,

$$(Fn)(x) = \lim_{a \to \infty} \frac{1}{2} \int_a^a n(y) e^{-ixy} dy,$$

where \( \text{l.i.m} \) denotes limit in the mean. Then we have

$$(F(Cn))(x) = i(\text{sgn } x)(Fn)(x).$$

Let us introduce the transformation

$$(C_\alpha f)(x) = (x - i)(Cn)(x)$$

for $f$ in $W^2$.

**Theorem 2.** The family \( \{P(t); 0 \leq t < \infty\} \) of linear operators on $W^2$ is a semigroup of class \((C_0)\). Its infinitesimal generator $A$ with domain $D(A)$ is given by the following form:

\[
D(A) = \{f \in W^2: (Cn)(x) \text{ is absolutely continuous and } (DCn)(x) \in L^2(R)\}
\]

and

\[
Af = DCGf
\]

for $f$ in $D(A)$.

If $f$ belongs to the domain $D([DCG]^2)$, then

\[
DCGDCGf = -D^2f.
\]

**Proof.** By [5. Theorem 1.1] it suffices to obtain the infinitesimal generator $A$ and its domain $D(A)$. From [5. (1.4)] we see that if $t$ is sufficiently small

\[
t^{-1}(P(t)f - f)(v) = (v-i)t^{-1}\left( -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(u)}{(u-i+it)(u-v+it)} du \right.
\]

\[
+ \frac{1}{2\pi i} \left. \int_{-\infty}^{\infty} \frac{f(u)}{(u-i-it)(u-v-it)} du \right) - \frac{f(v)}{v-i} + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(u)}{(u-i-it)(u-i+it)} du - n(v)
\]

\[
= (v-i)t^{-1}\left( -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{n(u)}{u-v+it} du + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{n(u)}{u-v-it} du - n(v) \right)
\]

\[
+ (v-i)t^{-1}\left( \frac{1}{2\pi i} \int_{-\infty}^{\infty} f(u) \left( \frac{1}{u-i} - \frac{1}{u-i+it} \right) du 
\]

\[
+ \frac{1}{2\pi i} \int_{-\infty}^{\infty} f(u) \left( \frac{1}{u-i-it} - \frac{1}{u-i} \right) du 
\]

\[
+ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(u)}{(u-i-it)(u-i+it)} du
\]
It is easily seen that

\[ \int_{-\infty}^{\infty} \frac{f(u)}{(u-i-it)(u-i+it)} \, du \rightarrow \int_{-\infty}^{\infty} g(u) \, du \quad \text{as} \quad t \rightarrow 0. \]  

Let

\[ g(t, u) = \frac{n(u)}{u-i+it}. \]

By [6. Proof of Theorem 101] we see that

\[ \left( \int_{-\infty}^{\infty} \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} g(t, u) \frac{du}{u-v+it} + \frac{1}{2\pi} \int_{-\infty}^{\infty} g(-t, u) \frac{du}{u-v-it} \right| \right)^{1/2} \]

\[ \leq \frac{1+M}{2} \left( \| g(t, \cdot) - g \|_2 + \| g(-t, \cdot) - g \|_2 \right) \rightarrow 0 \]  

as \( t \rightarrow 0 \), where \( \| \cdot \|_2 \) denotes the \( L^2 \)-norm and \( M \) is a constant number. On the other hand we see from [6. Theorem 92] that

\[ \left( \int_{-\infty}^{\infty} \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} g(u) \frac{du}{u-v+it} - \frac{1}{2\pi} \int_{-\infty}^{\infty} g(u) \frac{du}{u-v-it} \right| \right)^{1/2} \]

\[ \leq \left( \int_{-\infty}^{\infty} \left| g(u) \frac{du}{u-v+it} - (Cg)(v) \right|^2 \, dv \right)^{1/2} \rightarrow 0 \]  

as \( t \rightarrow 0 \). By (2.1)–(2.4) and by [3] we see that

\[ D(A) = \{ f \in W^2 : (Cn)(x) \text{ is absolutely continuous and } (DCn)(x) \in L^2(R) \} \]

and

\[ (Af)(v) = (v-i)(DCn)(v) + (v-i)(Cg)(v) + \frac{1}{\pi} \int_{-\infty}^{\infty} g(u) \, du \]

for \( f \in D(A) \). From [1. Pages 128–129] we obtain

\[ (Cn)(v) = (v-i)(Cg)(v) + \frac{1}{\pi} \int_{-\infty}^{\infty} g(u) \, du \quad \text{and} \quad Af = DCn f. \]

Let us prove the latter part. If \( f \) belongs to the domain \( D(A^2) \), it follows from \( f(x) = (x-i)n(x) \) that \( n(x), n'(x) \) are absolutely continuous and \( n'(x), n''(x) \in L^2(R) \). We obtain \( DCn = CDn \) from [3] and so
(DCGf)(v) = (v - i)(CDn)(v) + (Cn)(v).

By the inversion formula of the Hilbert transform and by [1. Pages 128–129] we have

\[(Cn(DCGf))(v) = (v - i)\left[ C(CDn) \right](v) + \left( v - i \right) C \left[ \frac{1}{(\cdot)} \right] (Cn)(\cdot) \]

= \(-(v - i)(Dn)(v) - n(v) + K(Cn),

where \( K(Cn) \) is a constant number. We lastly obtain

\[(DC^2DCGf)(v) = -2n'(v) - (v - i)n''(v) = -f''(v). \quad \text{Q. E. D.}\]

It is easy to show the following facts;

1. If \( f \) is a sufficiently regular function in \( L^2(R) \), it is shown in [3] that \( DCf = CDf \), but here it holds that

\[DCGf - CDf = \int_{-\infty}^{\infty} \frac{f(u)}{(u - i)^2} du.\]

2. The operator \( DCG \) is connected with the infinitesimal generator \( D^2 \) of \( T(t) \) by

\[DCGf = -(-D^2)^{1/2}f\]

for \( f \) in \( D(D^2) \). (See [7. Page 268])

References