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On Certain Connected Submanifolds of Euclidean Sphere

Kohei HATSUSE*

Let $\bar{M}^m$ be a simply connected, complete Riemannian manifold of non-positive constant curvature, that is, a Euclidean space $E^m$ or a hyperbolic space $H^m$. Let $M^n$, $n \geq 2$, be a differentiable manifold. When $f$ is an immersion of $M^n$ into $\bar{M}^m$, we can define a function $L_p$ on $M^n$ by

$$L_p(x) = d(p, f(x)) \quad x \in M^n,$$

where $d$ is the distance function in $\bar{M}^m$. For almost all $p \in \bar{M}^m$, $L_p$ is a Morse function on $M^n$. Nomizu and Rodriguez [5] proved the following

**Theorem A.** Let $M^n$ be a connected, complete Riemannian manifold isometrically immersed into $E^m$. If every Morse function $L_p$ on $M^n$ has index 0 or $n$ at any of its critical points, then $M^n$ is imbedded as a Euclidean $n$-subspace $E^n$ or a Euclidean $n$-sphere $S^n$.

This theorem includes that if $M^n$ is compact such that every Morse function $L_p$ has exactly two critical points, then $M^n$ is imbedded as a Euclidean $n$-sphere $S^n$. Cecil [2] proved the following

**Theorem B.** Let $M^n$ be a connected, compact manifold immersed into $H^m$. If every Morse function $L_p$ on $M^n$ has exactly two critical points, then $M^n$ is imbedded as a metric $n$-sphere $S^n$.

On the other hand, the author investigated in [3] connected, compact submanifolds $M^n$ of $E^m$ such that $C_n(L_p)=1$ or $C_0(L_p)=1$ for all Morse functions $L_p$ on $M^n$, where $C_k(L_p)$ is the number of critical points of $L_p$ of index $k$.

In this note, we deal with connected submanifolds $M^n$ of a Euclidean sphere $S^m$.

We shall mean $C^\infty$ differentiable by "differentiable" and an $n(\geq 2)$-dimensional differentiable manifold $M^n$ by a manifold $M^n$.

1. Preliminaries. Let $M^n$ be a connected manifold. Let $f$ be an immersion of $M^n$ into a Euclidean $m$-sphere $S^m$ about the origin of $E^{m+1}$, $m \geq n+1$. Since $f$ induces a Riemannian metric on $M^n$, $M^n$ satisfies the second axiom of countability ([4], p. 271). Therefore $K = \{ y \in S^m; y = f(x) \text{ or } -f(x) \text{ for } x \in M^n \}$

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has measure zero. For a point \( p \in S^m - K \), we define a function \( L_p \) on \( M^n \) by
\[
L_p(x) = d(p, f(x)) \quad x \in M^n,
\]
the distance in \( S^m \) from \( p \) to \( f(x) \). If \( S^m \) is a unit sphere, then a geodesic \( c(s) \) in \( S^m \), parametrized by arc length \( s \), such that \( c(0) = y \) and \( \dot{c}(0) = e \) is given by \( c(s) = y \cos s + e \sin s \). This implies
\[
\cos L_p(x) = \langle p, f(x) \rangle,
\]
where \( \langle , \rangle \) is the Euclidean inner product in \( E^{m+1} \). Therefore \( x \in M^n \) is a critical point of \( L_p \) if and only if \( p = f(x) \cos L_p(x) + e \sin L_p(x) \) and \( e \) is a unit vector normal to \( f(M^n) \) at \( f(x) \).

We denote by \( \mathbb{N}(M) \) the normal bundle to \( M^n \). The exponential mapping \( \exp: \mathbb{N}(M) \to S^m \) is given by \( \exp(x, se) = f(x) \cos s + e \sin s \) for \( (x, se) \in \mathbb{N}(M) \). If \( p = \exp(x, se) \) and the Jacobian \( \exp_\ast \) of \( \exp \) has nullity \( k > 0 \) at \((x, se)\), then we shall call \( p \) a focal point of \((M^n, x)\) of multiplicity \( k \). For \((x, e) \in \mathbb{N}(M)\), we denote by \( A_e \) a symmetric endomorphism of the tangent space \( T_x M^n \) which corresponds to the second fundamental form of \( M^n \).

We assume hereafter \( S^m \) is a unit sphere.

**Lemma 1.** A point \( p = \exp(x, s_0 e) \), \( 0 < s_0 < \pi \), is a focal point of \((M^n, x)\) of multiplicity \( k \) if and only if \( \cot s_0 = \lambda \) for an eigenvalue \( \lambda \) of \( A_e \) of multiplicity \( k \).

**Proof.** Let \((u^1, \ldots, u^n, U)\) be a coordinate system at \( x \in M^n \). Let \( e_1, \ldots, e_{m-n} \) be orthonormal vector fields normal to \( f(U) \) such that \( e_1(x) = e \). If \( x' \in U \) and \( v \) is a vector normal to \( f(U) \) at \( f(x') \), then we have
\[
v = s\left(\sqrt{1 - \sum_{i=2}^{m-n} (t^i)^2 e_i(x') + t^2 e_2(x') + \cdots + t^{m-n} e_{m-n}(x')}\right).
\]
This implies \((u^1, \ldots, u^n, s, t^2, \ldots, t^{m-n})\) are coordinates at \((x, s_0 e)\) in \( \mathbb{N}(M) \). By the definition of mapping \( \exp \), we obtain
\[
\exp_\ast(\partial/\partial s)_{(x, s_0 e)} = -f(x) \sin s_0 + e \cos s_0
\]
and
\[
\exp_\ast(\partial/\partial t^i)_{(x, s_0 e)} = e_i(x) \sin s_0.
\]
Let \( X = \sum_{i=1}^n \xi^i(\partial/\partial u^i)_{(x, s_0 e)} \). Since \( T_{(x, s_0 e)} \mathbb{N}(M) = T_x M^n \oplus E^{m-n} \), there exists \( Y \in T_x M^n \) and \((Y, 0) = X \). We assume the vector field \( e_1 \) normal to \( f(U) \) such that \( \nabla_{f_\ast(Y)} e_1 = -f_\ast(A_e Y) \), where \( \nabla \) is the Riemannian connection in \( S^m \). We denote by \( V(s) \) an infinitesimal variation of geodesic \( c(s) = f(x) \cos s + e \sin s \), \( 0 \leq s \leq s_0 \), such that \( V(0) = f_\ast(Y) \) and \( \nabla_\ast V = -f_\ast(A_e Y) \). Then we have
\[
\exp_\ast X = V(s_0).
\]
Therefore \( \exp_X X \) is the parallel displacement of \( f_*(Y) \cos s_0 - f_*(A_\varepsilon Y) \sin s_0 \) along the geodesic \( c(s) \) from \( f(x) \) to \( p \). This implies \( \exp_X X = 0 \) if and only if \( A_\varepsilon Y = Y \cot s_0 \), that is, \( \cot s_0 \) is an eigenvalue of \( A_\varepsilon \).

It is obvious that

\[
\exp_X X' = 0
\]

for \( X' \in T_{(x, \omega_0)} S^2 \) only if \( X' = \sum \xi_j (\partial/\partial u^j)_{(x, \omega_0)} \). Thus we have proved the lemma.

**Lemma 2.** Let \( p \in S^m - K \) and \( x \in M^n \). Then

1. \( x \) is a degenerate critical point of \( L_p \) if and only if \( \cot L_p(x) = \lambda \) for an eigenvalue \( \lambda \) of \( A_\varepsilon \).
2. If \( x \) is a non-degenerate critical point of \( L_p \), then the index of \( L_p \) at \( x \) is equal to the number of eigenvalues \( \lambda \) of \( A_\varepsilon \) such that \( \cot L_p(x) < \lambda \), counting multiplicities.

**Proof.** We choose a coordinate system \((u^1, \ldots, u^n, U)\) at \( x \in M^n \) such that \( \{(\partial/\partial u^1)_x, \ldots, (\partial/\partial u^n)_x\} \) is an orthonormal basis of \( T_x M^n \). We put \( y = f(x') \) for \( x' \in U \). Then we have

\[
- \frac{\partial^2 L_p}{\partial u^i \partial u^j} \sin L_p(x') - \frac{\partial L_p}{\partial u^i} \frac{\partial L_p}{\partial u^j} \cos L_p(x')
\]

\[
= \langle p, \frac{\partial^2 y}{\partial u^i \partial u^j} \rangle
\]

from \( \cos L_p(x') = \langle p, y \rangle \). Therefore if \( x \) is a critical point of \( L_p \), then

\[
- \frac{\partial^2 L_p}{\partial u^i \partial u^j}(x) \sin L_p(x)
\]

\[
= \langle p, \frac{\partial^2 y}{\partial u^i \partial u^j}(x) \rangle
\]

\[
= \langle f(x), \frac{\partial^2 y}{\partial u^i \partial u^j}(x) \rangle \cos L_p(x) + \langle e, \frac{\partial^2 y}{\partial u^i \partial u^j}(x) \rangle \sin L_p(x),
\]

because \( p = f(x) \cos L_p(x) + e \sin L_p(x) \). Since \( \langle y, \partial y/\partial u^i \rangle = 0 \), we have

\[
\langle f(x), \frac{\partial^2 y}{\partial u^i \partial u^j}(x) \rangle = - \delta_{ij}.
\]

On the other hand, if we write \( A_\varepsilon (\partial/\partial u^j)_x = \sum_{i=1}^n A_{\varepsilon ij}(x) (\partial/\partial u^i)_x \), then we have easily that

\[
\langle e, \frac{\partial^2 y}{\partial u^i \partial u^j}(x) \rangle = A_{\varepsilon ij}(x).
\]

Therefore we obtain
that is, the Hessian $H$ of $L_p$ at $x$ is given by

$$H(X, Y) = \langle (I \cot L_p(x) - A_x)X, Y \rangle$$

$X, Y \in T_xM^n$. Here $I$ denotes the identity mapping of $T_xM^n$ into itself. Thus the lemma follows.

By virtue of Lemmas 1 and 2, $L_p$ is a Morse function on $M^n$ for almost all $p \in S^n - K$. Furthermore, Lemmas 1 and 2 show the following

**Theorem 1.** (Index theorem for $L_p$) For $p \in S^n - K$, the index of $L_p$ at a non-degenerate critical point $x \in M^n$ is equal to the number of focal points of $(M^n, x)$ which lie on the minimizing geodesic $c(s) = f(x)\cos s + e\sin s, 0 \leq s \leq L_p(x)$, from $f(x)$ to $p$. Each focal point is counted with its multiplicity.

**Lemma 3.** If $x \in M^n$ is a non-degenerate critical point of $L_p$ of index $k$, then there exist sequences $\{x_a\}$ of points in $M^n$ and $\{p_a\}$ of points in $S^n - K$ such that

1. $\{x_a\}$ and $\{p_a\}$ converge to $x$ and $p$ respectively,
2. $L_{p_a}$ is a Morse function on $M^n$ and $x_a$ is a critical point of $L_{p_a}$ of index $k$.

**2. Main theorems**

**Theorem 2.** Let $M^n$ be a connected, complete Riemannian manifold isometrically immersed into $S^m$. If every Morse function $L_p$ on $M^n$ has index 0 or $n$ at each critical point of it, then $M^n$ is imbedded as a small $n$-sphere $S^n$.

**Proof.** Let $x \in M^n$ and let $e$ be a unit vector normal to $f(M^n)$ at $f(x)$. We assume $\lambda$ is the largest eigenvalue of $A_e$. If there exists the next largest eigenvalue $\mu$ of $A_e$, then $p = f(x)\cos L_p(x) + e\sin L_p(x), \lambda > \cot L_p(x) > \mu$, is not a focal point of $(M^n, x)$ from Lemma 1. Therefore $L_p$ has $x$ as a non-degenerate critical point. The index $k$ of $L_p$ at $x$ is equal to the multiplicity of eigenvalue $\lambda$ of $A_e$ from Theorem 1. There exists, from Lemma 3, a Morse function $L_q$ on $M^n$ which has a critical point $x'$ of index $k$. By the assumption, $L_q$ has index 0 or $n$ at $x'$. And so we have $k = n$ since $k$ cannot be 0. This shows that $A_e$ has only one eigenvalue $\lambda$ of multiplicity $n$, which implies $M^n$ is a totally umbilical submanifold of $E^{m+1}$, through the immersion $f$. Therefore $M^n$ is imbedded into $E^{m+1}$ as a Euclidean $n$-sphere ([1], p. 231). Hence $M^n$ is imbedded into $S^m$ as a small $n$-sphere $S^n$.

**Corollary.** Let $M^n$ be a connected, compact manifold immersed into $S^m$. If every Morse function $L_p$ on $M^n$ has exactly two critical points, then $M^n$ is imbedded as a small $n$-sphere $S^n$.

**Proof.** Since $M^n$ is compact, a Morse function $L_p$ on $M^n$ attains to the
maximal value at one critical point and the minimal value at the other. Therefore the corollary follows.

**Theorem 3.** Let $M^n$ be a connected, compact manifold immersed into $S^m$. Assume that

1. there exists an umbilical point $x \in M^n$,
2. $C_n(L_p) = 1$ for every Morse function $L_p$ on $M^n$.

Then $M^n$ is imbedded as a small $n$-sphere $S^n$.

**Proof.** If $x \in M^n$ is an umbilical point, then we have $A_e = \lambda(e)I$ for any vector $e$ normal to $f(M^n)$ at $f(x)$. Let $e_1, \ldots, e_{m-n}$ be orthonormal vectors normal to $f(M^n)$ at $f(x)$. Then, from Lemma 1, only one focal point $p_i$ of $(M^n, x)$ lies on each geodesic $c_i(s) = f(x) \cos s + e_i \sin s$, $0 \leq s < \pi$, where $1 \leq i \leq m - n$. The focal point $p_i$ is given by $p_i = f(x) \cos L_{p_i}(x) + e_i \sin L_{p_i}(x)$ and cot $L_{p_i}(x) = \lambda(e_i)$. For a point $p = f(x) \cos L_p(x) + e_i \sin L_p(x)$ such that $L_{p_i}(x) < L_p(x) < \pi$, $x$ is a non-degenerate critical point of $L_p$ of index $n$ from Theorem 1. There exist sequences $\{x_a\}$ of points in $M^n$ and $\{p_a\}$ of points in $S^m - K$ as in Lemma 3. Since $C_n(L_{p_a}) = 1$ from (2), each Morse function $L_{p_a}$ on $M^n$ attains to the maximal value at $x_a$. Therefore

$$L_p(x) = \lim_{a \to x} L_{p_a}(x_a) \geq \lim_{a \to x} L_{p_a}(x') = L_p(x')$$

for all $x' \in M^n$.

Let $\hat{c}_i(s) = f(x) \cos s - e_i \sin s$, $0 \leq s < \pi$. Then, from $\lambda(-e_i) = -\lambda(e_i)$, the antipodal point $-p_i$ of $p_i$ is only one focal point of $(M^n, x)$ along the geodesic $\hat{c}_i(s)$. Therefore if $q = f(x) \cos L_q(x) - e_i \sin L_q(x)$ such that $L_{-p_i}(x) < L_q(x) < \pi$, then $L_q$ attains to the maximal value at $x$. We have

$$L_{-q}(x') + L_{-q}(x') = \pi$$

for $x' \in M^n$. Hence $L_{-q}$ attains to the minimal value at $x$. The point $-q$ can be written as $-q = f(x) \cos L_{-q}(x) + e_i \sin L_{-q}(x)$ such that $0 < L_{-q}(x) < L_p(x)$. Thus each $L_{p_i}$ is a constant function on $M^n$, and hence there exists in $E^{m+1}$ a linear subvariety $E^m_1$ perpendicular to $p_i$ such that $f(M^n) \subseteq E^m_1$. Since $e_1, \ldots, e_{m-n}$ are orthonormal, $p_1, \ldots, p_{m-n}$ are linearly independent. Therefore $E^m_1 \cap \cdots \cap E^m_{m-n}$ is a linear subvariety $E^{m+1}$ in $E^{m+1}$. We know that $S^m \cap E^{m+1}$ is a Euclidean $n$-sphere $S^n$. The mapping $f : M^n \to S^n$ is an immersion. Since $S^n$ is simply connected from $n \geq 2$, $f$ is a diffeomorphism. Therefore $M^n$ is imbedded as a small $n$-sphere $S^n$.

**Theorem 4.** Let $M^n$ be a connected, compact manifold immersed into $S^m$. Assume that

1. there exists an umbilical point $x \in M^n$,
2. $C_0(L_p) = 1$ for every Morse function $L_p$ on $M^n$.

Then $M^n$ is imbedded as a small $n$-sphere $S^n$.

**Proof.** If $x \in M^n$ is an umbilical point, then we have $A_e = \lambda(e)I$ for a unit vector $e$ normal to $f(M^n)$ at $f(x)$. Therefore, from Lemma 1, $p' = f(x) \cos L_p(x) + e \sin L_p(x)$ such that $\cot L_p(x) = \lambda(e)$ is only one focal point of $(M^n, x)$ along a geodesic $c(s) = f(x) \cos s + e \sin s$, $0 \leq s < \pi$. For a point $p = f(x) \cos L_p(x) + e \sin L_p(x)$ such that $0 < L_p(x) < L_p(x)$, $x$ is a non-degenerate critical point of $L_p$ of index 0 from Theorem 1. There exist sequences $\{x_a\}$ of points in $M^n$ and $\{p_a\}$ of points in $S^m - K$ as in Lemma 3. Since $C_0(L_{p_a}) = 1$ from (2), each Morse function $L_{p_a}$ attains to the minimal value at $x_a$. Therefore

$$L_p(x) = \lim_{a} L_{p_a}(x_a) \leq \lim_{a} L_{p_a}(x') = L_p(x')$$

for all $x' \in M^n$. Similarly, if $q = f(x) \cos L_q(x) - e \sin L_q(x)$ such that $0 < L_q(x) < L_{-q}(x)$, then $L_q$ attains to the minimal value at $x$, which implies $L_{-q}$ attains to the maximal value at $x$. Thus $L_p$ is a constant function on $M^n$. Therefore we can find a small $n$-sphere $S^n$ such that $f(M^n) \subset S^n$. Since $f: M^n \to S^n$ is an immersion and $S^n$ is simply connected from $n \geq 2$, $M^n$ is imbedded as a small $n$-sphere $S^n$.

**References**


