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On Certain Compact Submanifolds of Euclidean Space

Kohei Hatsuse

Let $M^n$ be a differentiable manifold. When $f$ is an immersion of $M^n$ into a Euclidean space $E^m$, we can define a function $L_p$ on $M^n$ by

$$L_p(x) = \langle p - f(x), p - f(x) \rangle \quad x \in M^n$$

where $\langle , \rangle$ denotes the Euclidean inner product in $E^m$. For almost all points $p \in E^m$, $L_p$ is a Morse function on $M^n$. Nomizu and Rodriguez [2] proved the following

**Theorem.** Let $M^n, n \geq 2$, be a compact connected manifold immersed into $E^m$. If every Morse function $L_p$ on $M^n$ has exactly two critical points, then $M^n$ is imbedded as a Euclidean n-sphere $S^n$.

If a Morse function $L_p$ on $M^n$ has exactly two critical points, then both of the numbers $C_0(L_p)$ of critical points of index zero and $C_n(L_p)$ of critical points of index $n$ are equal to one. The purpose of this note is to investigate $M^n$ such that $C_0(L_p) = 1$ or $C_n(L_p) = 1$ for every Morse function $L_p$ on $M^n$.

We shall mean $C^\infty$ differentiable by "differentiable" and an $n(\geq 2)$-dimensional differentiable manifold $M^n$ by a manifold $M^n$.

1. Lemmas

The notions and notations will follow from Milnor [1]. Let $f : M^n \rightarrow E^m$ be an immersion and let $\nabla(M)$ be the normal bundle to $M^n$. If $L_p$ is a Morse function on $M^n$, then $p \in E^m$ is not a focal point of $M^n$. Let $x \in M^n$ and let $e(x) \in \nabla(M)$, that is, a vector normal to $f(M^n)$ at $f(x)$. We denote by $A_{e(x)}$ a symmetric linear transformation of a tangent space $T_xM^n$ into itself which corresponds to the second fundamental form of $M^n$. If $x$ is a critical point of Morse function $L_p$ on $M^n$ and if $p = f(x) + te(x), t > 0$, then the index at $x$ is equal to the number of eigenvalues $\alpha$ of $A_{e(x)}$ such that $0 < 1/\alpha < t$, counting multiplicities.

We shall identify $x \in M^n$ with $f(x)$ if there is no confusion. We begin with the following

**Lemma 1.** Let $M^n$ be a manifold immersed into $E^{n+1}$. Then $M^n$ is convex if and only if there exists a unit vector $e(x)$ normal to $M^n$ at each $x \in M^n$ and

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for all $t > 0$ where $p(t, x) = x + te(x)$.

**Proof.** Let $x \in M^n$. If $M^n$ is convex then there exists a unit vector $e(x)$ normal to $M^n$ at $x$ such that $<e(x), y - x> \leq 0$ for all $y \in M^n$. Let $t > 0$ then

$$L_{p(t,x)}(x) = 2t <e(x), x - y> + <x - y, x - y> \geq 0.$$  

Therefore $L_{p(t,x)}(x) \leq L_{p(t,x)}(y)$. Conversely, we suppose that there exists a unit vector $e(x)$ normal to $M^n$ and $L_{p(t,x)}(x) \leq L_{p(t,x)}(y)$ for all $t > 0$. Then $<e(x), y - x> \leq 0$ for all $y \in M^n$. Because, if there exists a point $y_0 \in M^n$ such that $<e(x), y_0 - x> > 0$, then

$$L_{p(t,x)}(x) > L_{p(t,x)}(y_0)$$  

for $t > <x - y_0, x - y_0>/2 <e(x), y_0 - x>$. Therefore $M^n$ is convex.

**Lemma 2.** Let $M^n$ be a convex manifold immersed into $E^{n+1}$. Let $e(x)$ be a unit vector normal to $M^n$ at $x$ as in Lemma 1. Then every eigenvalue of $A_{e(x)}$ is non-positive.

**Proof.** The mapping $e : M^n \ni x \mapsto e(x) \in \mathbb{L}(M)$ defines a differentiable vector field along $M^n$. We suppose that $A_{e(x)}$ has a positive eigenvalue. Let $\alpha$ be the largest positive eigenvalue of $A_{e(x)}$ whose multiplicity is $k$, and let $\beta$ be the next largest positive eigenvalue of $A_{e(x)}$. Take $t > 0$ such that $1/\alpha < t < 1/\beta$ (if $\alpha$ is the only positive eigenvalue, just considered $1/\alpha < t$). Then, from Lemma of [2], there exists a point $x' \in M^n$ and it is a critical point of a Morse function $L_{p(t,x)}$ on $M^n$ of index $k$. On the other hand, Lemma 1 implies the index at $x'$ is zero. Therefore Lemma is proved.

**Lemma 3.** Let $M^n$ be a compact connected manifold immersed into $E^{n+1}$. If $M^n$ is convex then $M^n$ is diffeomorphic to $n$-sphere $S^n$.

**Proof.** Since $M^n$ is compact, there exists an open ball $D$ of radius $r$ about the origin of $E^{n+1}$ such that $M^n \subset D$. The boundary of $D$ is an $n$-sphere $S^n$. Every ray $p(t, x) = x + te(x)$, $t > 0$, starting from $x \in M^n$ meets $S^n$ at only one point $p(t(x), x)$. If we define a mapping $\phi : M^n \rightarrow S^n$ by

$$\phi(x) = p(t(x), x) \quad x \in M^n$$  

then $\phi$ is differentiable since

$$t(x) = -<e(x), x> + \{r^2 - <x, x> + <e(x), x>^2\}^{1/2}.$$  

We denote by $\exp$ the exponential mapping of $\mathbb{L}(M)$ into $E^{n+1}$ and denote by $\pi$ the projection of $\mathbb{L}(M)$ into $M^n$. Then, by the definition of $\phi$, we obtain $\phi(x) = \exp t(x)e(x)$. Let $(v^1, \ldots, v^{n+1}, U)$ be a cubical coordinate system centered
at \( \phi(x) \) in \( E^{n+1} \) such that \( U \cap S^n \) is an \( n \)-dimensional slice defined by \( v^{n+1} = v^{n+1}(\phi(x)) \). Let \((u^1, \ldots, u^n, V)\) be a coordinate system at \( x \) in \( M^n \) such that

\[
\phi(V) \subseteq U \cap S^n.
\]

There exists a function \( u \) in \( \mathcal{U}(M) \), and \( u^1 \circ \pi, \ldots, u^n \circ \pi, u \) form a system of coordinates at \( t(x) e(x) \). By virtue of Lemma 2, \( \exp \) is regular at \( t(x) e(x) \). Therefore, \( \phi \) is a univalent mapping of \( V \) into \( U \cap S^n \), and \( u^1 \circ \phi^{-1} = u^1 \circ \pi \circ \exp^{-1} \mid \phi(V) \) is a differentiable function of \( v^1, \ldots, v^n \). This implies \( \phi \) is an immersion. Consequently, \( \phi \) is a diffeomorphism of \( M^n \) into \( S^n \) since \( S^n \) is simply connected when \( n \geq 2 \).

Lemma of Nomizu and Rodríguez [2] can be stated as in the following

**Lemma** ([2], p. 119). Let \( M^n \) be a manifold immersed into \( E^m \). Let \( p \in E^m \) and assume that \( L_p \) has a non-degenerate critical point \( x \in M^n \) of index \( k \). Then there exist sequences \( \{x_h\} \) of points in \( M^n \) and \( \{p_h\} \) of points in \( E^m \) such that

1. \( \{x_h\} \) and \( \{p_h\} \) converge to \( x \) and \( p \) respectively,
2. \( L_{p_h} \) is a Morse function on \( M^n \), and \( x_h \) is a critical point of \( L_{p_h} \) of index \( k \).

### 2. Theorems

**Lemma 4.** Let \( M^n \) be a manifold immersed into \( E^m \), \( m > n+1 \). Let \( x \in M^n \) and let \( e(x) \) be a unit vector normal to \( M^n \) at \( x \). We put \( p(t, x) = x + te(x) \) when \( t > 0 \) and \( q(s, x) = x + se(x) \) when \( s < 0 \). Then \( M^n \) belongs to a linear variety \( E^{m-1} \) if one of the following properties holds for every unit normal vector \( e(x) \) at \( x \):

(a) \[
L_{p(t, x)}(x) \leq L_{p(t, x)}(y) \quad y \in M^n
\]

for all \( t > 0 \).

(b) \[
L_{q(s, x)}(x) \leq L_{q(s, x)}(y) \quad y \in M^n
\]

for all \( s < 0 \).

**Proof.** Let \( S \) be a set of all unit vectors \( e(x) \) normal to \( M^n \) at \( x \). Then \( S \) can be considered as a Euclidean \( (m-n-1) \)-sphere. We suppose that there exists a unit vector \( e'(x) \) normal to \( M^n \) at \( x \) such that (a) does not hold for it. If we denote by \( B \) a set of such unit vectors \( e'(x) \), then \( B \) is an open subset of \( S \). Because, if \( B \) is not an open subset of \( S \), there exists a sequence \( \{e_k(x)\} \) of unit vectors in \( S - B \) which converges to a suitable \( e'(x) \in B \). For each \( e_j(x) \), (a) holds. Therefore,

\[
L_{p'(t, x)}(x) = \lim_{k} L_{p_k(t, x)}(x)
\]

\[
\leq \lim_{k} L_{p_k(t, x)}(y) = L_{p'(t, x)}(y)
\]
for all \( t > 0 \) where \( p'(t, x) = x + te'(x) \) and \( p(t, x) = x + te_\lambda(x) \). This contradicts \( e'(x) \in B \).

When \( e'(x) \in B \), (a) holds for \(-e'(x)\) and hence \( B \equiv S \). Let \( e(x) \) be a boundary point of \( B \). Then there exists a sequence \( \{e_\mu(x)\} \) of unit vectors in \( B \) which converges to \( e(x) \). For each \( e_\mu(x) \), (b) holds. Therefore

\[
L_{q(s, x)}(x) = \lim_{\mu} L_{q_\mu(s, x)}(x) 
\leq \lim_{\mu} L_{q_\mu(s, x)}(y) = L_{q(s, x)}(y)
\]

for all \( s < 0 \) where \( q_\mu(s, x) = x + se_\mu(x) \). Since \( B \equiv S \), we obtain \( e(x) \notin B \). These imply both of (a) and (b) hold for \( e(x) \). Therefore

\[
< e(x), y - x > = 0
\]

for all \( y \in M^n \). Thus \( M^n \) belongs to a linear variety \( E^{n-1} \) perpendicular to \( e(x) \).

**Theorem 1.** Let \( M^n \) be a compact connected manifold immersed into \( E^n \). Let \( e(x) \in \perp(M) \) and let \( \alpha_1, \ldots, \alpha_n \) be eigenvalues of \( A_{e(x)} \). Assume that

1. \( \alpha_i \alpha_j \geq 0 \) (i, j = 1, ..., n) for every \( e(x) \in \perp(M) \),
2. \( C_n(L_p) = 1 \) for every Morse function \( L_p \) on \( M^n \).

Then \( M^n \) is diffeomorphic to \( n \)-sphere \( S^n \).

**Proof.** Let \( e(x) \in \perp(M) \) be a unit vector at \( x \in M^n \). We suppose that eigenvalues \( \alpha_1, \ldots, \alpha_n \) of \( A_{e(x)} \) are non-positive. Let \( t > 0 \). Then \( x \) is a non-degenerate critical point of \( L_{p(t, x)} \) of index zero. There exist sequences \( \{x_\mu \} \) of points in \( M^n \) and \( \{p_\mu \} \) of points in \( E^n \) as in Lemma of [2]. Since \( C_n(L_{p_\mu}) = 1 \) from (2), Morse function \( L_{p_\mu} \) attains to minimal value at exactly one point \( x_\mu \). Therefore

\[
L_{p(t, x)}(x) = \lim_{\mu} L_{p_\mu}(x_\mu) \leq \lim_{\mu} L_{p_\mu}(y) = L_{p(t, x)}(y)
\]

for all \( t > 0 \). From (1), this implies one of the properties (a) and (b) in Lemma 4 holds for every unit vector \( e(x) \) normal to \( M^n \). According to Lemma 4, \( M^n \) belongs to a linear variety \( E^{n+1} \). When \( M^n \subset E^{n+1} \), the properties (a) and (b) for every unit vector \( e(x) \in \perp(M) \) imply \( M^n \) is immersed into \( E^{n+1} \) as a convex hypersurface from Lemma 1. Thus, from Lemma 3, \( M^n \) is diffeomorphic to \( n \)-sphere \( S^n \).

**Theorem 2.** Let \( M^n \) be a compact connected manifold immersed into \( E^n \). Let \( e(x) \in \perp(M) \) and let \( \alpha_1, \ldots, \alpha_n \) be eigenvalues of \( A_{e(x)} \). We denote by \( N \) a set of \( e(x) \in \perp(M) \) such that \( \alpha_i \alpha_j > 0 \) (i, j = 1, ..., n). Assume that

1. \( \perp(M) = \overline{N} \),
2. \( C_n(L_p) = 1 \) for every Morse function \( L_p \) on \( M^n \).
Then $M^n$ is diffeomorphic to $n$-sphere $S^n$.

**Proof.** Let $e(x) \in \perp(M)$ be a unit vector at $x \in M^n$. We suppose that eigenvalues $a_1, \ldots, a_n$ of $A_{e(x)}$ are positive. Let $a_1$ be the smallest eigenvalue of $A_{e(x)}$. If $t > 1/a_1$ then $x$ is a non-degenerate critical point of $L_{p(x)}$ of index $n$. There exist sequences $\{x_\lambda\}$ of points in $M^n$ and $\{p_\lambda\}$ of points in $E^n$ as in Lemma of [2]. Since $C_n(L_{p_\lambda}) = 1$ from (2), Morse function $L_{p_\lambda}$ attains to maximal value at exactly one point $x_\lambda$. Therefore

$$L_{p(t,x)}(x) = \lim_{\lambda} L_{p_\lambda}(x_\lambda) \geq \lim_{\lambda} L_{p_\lambda}(y) = L_{p(t,x)}(y)$$

and hence $<e(x), x - y> \leq 0$ for all $y \in M^n$. If $s < 0$ then we obtain

$$L_{q(t,x)}(x) - L_{q(s,x)}(y) = -2s <e(x), x - y> - <x - y, x - y> \leq 0$$

for all $y \in M^n$. By the definition of $N$, this implies one of the properties (a) and (b) in Lemma 4 holds for every unit vector $e(x) \in N$.

When $e(z) \in \perp(M)$ is a unit vector at $z \in M^n$, there exists a sequence $\{e_\mu(x_\mu)\}$ of unit vectors in $N$ which converges to $e(z)$. If (a) holds for all $e_\mu(x_\mu)$, then

$$L_{p(t,z)}(z) = \lim_{\mu} L_{p_\mu(t,x_\mu)}(x_\mu) \leq \lim_{\mu} L_{p_\mu(t,x_\mu)}(y) = L_{p(t,z)}(y)$$

for all $t > 0$ and $y \in M^n$ where $p_\mu(t,x_\mu) = x_\mu + te_\mu(x_\mu)$. Similarly, if (b) holds for all $e_\mu(x_\mu)$ then (b) holds for $e(z)$. Thus one of the properties (a) and (b) in Lemma 4 holds for every unit vector $e(x)$ normal to $M^n$. According to Lemmas 1 and 4, $M^n$ belongs to a linear variety $E^{n+1}$, and it is immersed into $E^{n+1}$ as a convex hypersurface. Consequently, from Lemma 3, $M^n$ is diffeomorphic to $n$-sphere $S^n$.

**References**
