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Title
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Citation
Bulletin of the Faculty of Science, Ibaraki University. Series A, Mathematics, 9: 55-59

Issue Date
1977

URL
http://hdl.handle.net/10109/2885

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On Certain Compact Submanifolds of Euclidean Space

Kohei Hatsuse

Let $M^n$ be a differentiable manifold. When $f$ is an immersion of $M^n$ into a Euclidean space $E^m$, we can define a function $L_p$ on $M^n$ by

$$L_p(x) = <p - f(x), p - f(x)> \quad x \in M^n$$

where $< , >$ denotes the Euclidean inner product in $E^m$. For almost all points $p \in E^m$, $L_p$ is a Morse function on $M^n$. Nomizu and Rodriguez [2] proved the following

**Theorem.** Let $M^n, n \geq 2$, be a compact connected manifold immersed into $E^m$. If every Morse function $L_p$ on $M^n$ has exactly two critical points, then $M^n$ is imbedded as a Euclidean $n$-sphere $S^n$.

If a Morse function $L_p$ on $M^n$ has exactly two critical points, then both of the numbers $C_0(L_p)$ of critical points of index zero and $C_n(L_p)$ of critical points of index $n$ are equal to one. The purpose of this note is to investigate $M^n$ such that $C_0(L_p) = 1$ or $C_n(L_p) = 1$ for every Morse function $L_p$ on $M^n$.

We shall mean $C^\infty$ differentiable by "differentiable" and an $n(\geq 2)$-dimensional differentiable manifold $M^n$ by a manifold $M^n$.

1. Lemmas

The notions and notations will follow from Milnor [1]. Let $f: M^n \rightarrow E^m$ be an immersion and let $\perp(M)$ be the normal bundle to $M^n$. If $L_p$ is a Morse function on $M^n$, then $p \in E^m$ is not a focal point of $M^n$. Let $x \in M^n$ and let $e(x) \in \perp(M)$, that is, a vector normal to $f(M^n)$ at $f(x)$. We denote by $A_{e(x)}$ a symmetric linear transformation of a tangent space $T_xM^n$ into itself which corresponds to the second fundamental form of $M^n$. If $x$ is a critical point of Morse function $L_p$ on $M^n$ and if $p = f(x) + te(x)$, $t > 0$, then the index at $x$ is equal to the number of eigenvalues $\alpha$ of $A_{e(x)}$ such that $0 < 1/\alpha < t$, counting multiplicities.

We shall identify $x \in M^n$ with $f(x)$ if there is no confusion. We begin with the following

**Lemma 1.** Let $M^n$ be a manifold immersed into $E^{n+1}$. Then $M^n$ is convex if and only if there exists a unit vector $e(x)$ normal to $M^n$ at each $x \in M^n$ and

Received January 20, 1977.

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for all \( t > 0 \) where \( p(t, x) = x + te(x) \).

**Proof.** Let \( x \in M^n \). If \( M^n \) is convex then there exists a unit vector \( e(x) \) normal to \( M^n \) at \( x \) such that \( \langle e(x), y - x \rangle \leq 0 \) for all \( y \in M^n \). Let \( t > 0 \) then

\[
L_{p(t,x)}(y) - L_{p(t,x)}(x) = 2t \langle e(x), y - x \rangle + \langle x - y, x - y \rangle \geq 0.
\]

Therefore \( L_{p(t,x)}(x) \leq L_{p(t,x)}(y) \). Conversely, we suppose that there exists a unit vector \( e(x) \) normal to \( M^n \) and \( L_{p(t,x)}(x) \leq L_{p(t,x)}(y) \) for all \( t > 0 \). Then \( \langle e(x), y - x \rangle \leq 0 \) for all \( y \in M^n \). Because, if there exists a point \( y_0 \in M^n \) such that \( \langle e(x), y_0 - x \rangle > 0 \), then

\[
L_{p(t,x)}(x) > L_{p(t,x)}(y_0)
\]

for \( t > \frac{\langle x - y_0, x - y_0 \rangle}{2 \langle e(x), y_0 - x \rangle} \). Therefore \( M^n \) is convex.

**Lemma 2.** Let \( M^n \) be a convex manifold immersed into \( E^{n+1} \). Let \( e(x) \) be a unit vector normal to \( M^n \) at \( x \) as in Lemma 1. Then every eigenvalue of \( A_{e(x)} \) is non-positive.

**Proof.** The mapping \( e: M^n \ni x \mapsto e(x) \in \perp(M) \) defines a differentiable vector field along \( M^n \). We suppose that \( A_{e(x)} \) has a positive eigenvalue. Let \( \alpha \) be the largest positive eigenvalue of \( A_{e(x)} \) whose multiplicity is \( k \), and let \( \beta \) be the next largest positive eigenvalue of \( A_{e(x)} \). Take \( t > 0 \) such that \( 1/\alpha < t < 1/\beta \) (if \( \alpha \) is the only positive eigenvalue, just considered \( 1/\alpha < t \)). Then, from Lemma of [2], there exists a point \( x' \in M^n \) and it is a critical point of a Morse function \( L_{p(t,x)} \) on \( M^n \) of index \( k \). On the other hand, Lemma 1 implies the index at \( x' \) is zero. Therefore Lemma is proved.

**Lemma 3.** Let \( M^n \) be a compact connected manifold immersed into \( E^{n+1} \). If \( M^n \) is convex then \( M^n \) is diffeomorphic to \( n \)-sphere \( S^n \).

**Proof.** Since \( M^n \) is compact, there exists an open ball \( D \) of radius \( r \) about the origin of \( E^{n+1} \) such that \( M^n \subset D \). The boundary of \( D \) is an \( n \)-sphere \( S^n \). Every ray \( p(t, x) = x + te(x), t > 0 \), starting from \( x \in M^n \) meets \( S^n \) at only one point \( p(t(x), x) \). If we define a mapping \( \phi: M^n \rightarrow S^n \) by

\[
\phi(x) = p(t(x), x) \quad x \in M^n
\]

then \( \phi \) is differentiable since

\[
t(x) = -\langle e(x), x \rangle + \{r^2 - \langle x, x \rangle + \langle e(x), x \rangle^2 \}^{1/2}.
\]

We denote by \( \exp \) the exponential mapping of \( \perp(M) \) into \( E^{n+1} \) and denote by \( \pi \) the projection of \( \perp(M) \) into \( M^n \). Then, by the definition of \( \phi \), we obtain \( \phi(x) = \exp t(x)e(x) \). Let \( (v^1, \ldots, v^{n+1}, U) \) be a cubical coordinate system centered
at $\phi(x)$ in $E^{n+1}$ such that $U \cap S^n$ is an $n$-dimensional slice defined by $v^{n+1}=v^{n+1}(\phi(x))$. Let $(u_1, \ldots, u_n, V)$ be a coordinate system at $x$ in $M^n$ such that $\phi(V) \subset U \cap S^n$. There exists a function $u$ in $\mathbb{L}(M)$, and $u^1 \circ \pi, \ldots, u^n \circ \pi, u$ form a system of coordinates at $t(x)e(x)$. By virtue of Lemma 2, $\exp$ is regular at $t(x)e(x)$. Therefore, $\phi$ is a univalent mapping of $V$ into $U \cap S^n$, and $u^1 \circ \phi^{-1} = u^1 \circ \pi \circ \exp^{-1}|\phi(V)$ is a differentiable function of $v^1, \ldots, v^n$. This implies $\phi$ is an immersion. Consequently, $\phi$ is a diffeomorphism of $M^n$ into $S^n$ since $S^n$ is simply connected when $n \geq 2$.

Lemma of Nomizu and Rodríguez [2] can be stated as in the following

**Lemma** ([2], p. 119). Let $M^n$ be a manifold immersed into $E^m$. Let $p \in E^m$ and assume that $L_p$ has a non-degenerate critical point $x \in M^n$ of index $k$. Then there exist sequences $\{x_\alpha\}$ of points in $M^n$ and $\{p_\alpha\}$ of points in $E^m$ such that

1. $\{x_\alpha\}$ and $\{p_\alpha\}$ converge to $x$ and $p$ respectively,
2. $L_{p_\alpha}$ is a Morse function on $M^n$, and $x_\alpha$ is a critical point of $L_{p_\alpha}$ of index $k$.

2. Theorems

**Lemma 4.** Let $M^n$ be a manifold immersed into $E^m$, $m > n + 1$. Let $x \in M^n$ and let $e(x)$ be a unit vector normal to $M^n$ at $x$. We put $p(t, x)=x+te(x)$ when $t > 0$ and $q(s, x)=x+se(x)$ when $s < 0$. Then $M^n$ belongs to a linear variety $E^{m-1}$ if one of the following properties holds for every unit normal vector $e(x)$ at $x$:

(a) $L_{p(t, x)}(x) \leq L_{p(t, x)}(y)$ for all $t > 0$.

(b) $L_{q(s, x)}(x) \leq L_{q(s, x)}(y)$ for all $s < 0$.

**Proof.** Let $S$ be a set of all unit vectors $e(x)$ normal to $M^n$ at $x$. Then $S$ can be considered as a Euclidean $(m-n-1)$-sphere. We suppose that there exists a unit vector $e'(x)$ normal to $M^n$ at $x$ such that (a) does not hold for it. If we denote by $B$ a set of such unit vectors $e'(x)$, then $B$ is an open subset of $S$. Because, if $B$ is not an open subset of $S$, there exists a sequence $\{e_\alpha(x)\}$ of unit vectors in $S-B$ which converges to a suitable $e'(x) \in B$. For each $e_\alpha(x)$, (a) holds. Therefore

$$L_{p'(t, x)}(x) = \lim_{\alpha} L_{p_{\alpha}(t, x)}(x) \leq \lim_{\alpha} L_{p_{\alpha}(t, x)}(y) = L_{p'(t, x)}(y)$$
for all $t > 0$ where $p'(t, x) = x + te'(x)$ and $p_\alpha(t, x) = x + te_\alpha(x)$. This contradicts $e'(x) \in B$.

When $e'(x) \in B$, (a) holds for $-e'(x)$ and hence $B \cong S$. Let $e(x)$ be a boundary point of $B$. Then there exists a sequence $\{e_\alpha(x)\}$ of unit vectors in $B$ which converges to $e(x)$. For each $e_\alpha'(x)$, (b) holds. Therefore

$$L_{q(s, x)}(x) = \lim_{\mu} L_{q_\mu'(s, x)}(x)$$

$$\leq \lim_{\mu} L_{q_\mu'(s, x)}(y) = L_{q(s, x)}(y)$$

for all $s < 0$ where $q_\mu'(s, x) = x + se'_\mu(x)$. Since $B \cong S$, we obtain $e(x) \notin B$. These imply both of (a) and (b) hold for $e(x)$. Therefore

$$<e(x), y-x> = 0$$

for all $y \in M^n$. Thus $M^n$ belongs to a linear variety $E^{n-1}$ perpendicular to $e(x)$.

**Theorem 1.** Let $M^n$ be a compact connected manifold immersed into $E^m$. Let $e(x) \in \perp(M)$ and let $\alpha_1, \ldots, \alpha_n$ be eigenvalues of $A_{e(x)}$. Assume that

1. $\alpha_i \alpha_j \geq 0$ ($i, j = 1, \ldots, n$) for every $e(x) \in \perp(M)$,

2. $C_0(L_p) = 1$ for every Morse function $L_p$ on $M^n$.

Then $M^n$ is diffeomorphic to $n$-sphere $S^n$.

**Proof.** Let $e(x) \in \perp(M)$ be a unit vector at $x \in M^n$. We suppose that eigenvalues $\alpha_1, \ldots, \alpha_n$ of $A_{e(x)}$ are non-positive. Let $t > 0$. Then $x$ is a non-degenerate critical point of $L_{p(t, x)}$ of index zero. There exist sequences $\{x_\lambda\}$ of points in $M^n$ and $\{p_\lambda\}$ of points in $E^m$ as in Lemma of [2]. Since $C_0(L_{p_\lambda}) = 1$ from (2), Morse function $L_{p_\lambda}$ attains to minimal value at exactly one point $x_\lambda$. Therefore

$$L_{p(t, x)}(x) = \lim_{\lambda} L_{p_\lambda}(x) \leq \lim_{\lambda} L_{p_\lambda}(y) = L_{p(t, x)}(y)$$

for all $t > 0$. From (1), this implies one of the properties (a) and (b) in Lemma 4 holds for every unit vector $e(x)$ normal to $M^n$. According to Lemma 4, $M^n$ belongs to a linear variety $E^{n+1}$. When $M^n \subset E^{n+1}$, the properties (a) and (b) for every unit vector $e(x) \in \perp(M)$ imply $M^n$ is immersed into $E^{n+1}$ as a convex hypersurface from Lemma 1. Thus, from Lemma 3, $M^n$ is diffeomorphic to $n$-sphere $S^n$.

**Theorem 2.** Let $M^n$ be a compact connected manifold immersed into $E^m$. Let $e(x) \in \perp(M)$ and let $\alpha_1, \ldots, \alpha_n$ be eigenvalues of $A_{e(x)}$. We denote by $N$ a set of $e(x) \in \perp(M)$ such that $\alpha_i \alpha_j > 0$ ($i, j = 1, \ldots, n$). Assume that

1. $\perp(M) = \overline{N}$,

2. $C_n(L_p) = 1$ for every Morse function $L_p$ on $M^n$. 

Then $M^n$ is diffeomorphic to $n$-sphere $S^n$.

Proof. Let $e(x) \in \mathbb{L}(M)$ be a unit vector at $x \in M^n$. We suppose that eigenvalues $\lambda_1, \ldots, \lambda_n$ of $A_{e(x)}$ are positive. Let $\lambda_1$ be the smallest eigenvalue of $A_{e(x)}$. If $t > 1/\lambda_1$ then $x$ is a non-degenerate critical point of $L_{p(t,x)}$ of index $n$. There exist sequences $\{x_\lambda\}$ of points in $M^n$ and $\{p_\lambda\}$ of points in $E^n$ as in Lemma of [2]. Since $C_n(L_{p_\lambda}) = 1$ from (2), Morse function $L_{p_\lambda}$ attains to maximal value at exactly one point $x_\lambda$. Therefore

$$L_{p(t,x)}(x) = \lim_{\lambda} L_{p_\lambda}(x_\lambda) \geq \lim_{\lambda} L_{p_\lambda}(y) = L_{p(t,x)}(y)$$

and hence $<e(x), x - y> \leq 0$ for all $y \in M^n$. If $s < 0$ then we obtain

$$L_{q(s,x)}(x) - L_{q(s,x)}(y) = -2s <e(x), x - y> -<x - y, x - y> \leq 0$$

for all $y \in M^n$. By the definition of $N$, this implies one of the properties (a) and (b) in Lemma 4 holds for every unit vector $e(x) \in N$.

When $e(z) \in \mathbb{L}(M)$ is a unit vector at $z \in M^n$, there exists a sequence $\{\mu(x_\mu)\}$ of unit vectors in $N$ which converges to $e(z)$. If (a) holds for all $\mu(x_\mu)$, then

$$L_{p(t,z)}(z) = \lim_{\mu} L_{p_\mu(t,x_\mu)}(x_\mu) \leq \lim_{\mu} L_{p_\mu(t,x_\mu)}(y) = L_{p(t,z)}(y)$$

for all $t > 0$ and $y \in M^n$ where $p_\mu(t,x_\mu) = x_\mu + t\mu(x_\mu)$. Similarly, if (b) holds for all $\mu(x_\mu)$ then (b) holds for $e(z)$. Thus one of the properties (a) and (b) in Lemma 4 holds for every unit vector $e(x)$ normal to $M^n$. According to Lemmas 1 and 4, $M^n$ belongs to a linear variety $E^{n+1}$, and it is immersed into $E^{n+1}$ as a convex hypersurface. Consequently, from Lemma 3, $M^n$ is diffeomorphic to $n$-sphere $S^n$.

References
