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On the Integral Dependence, the Henselian Property in Power Series Rings

Ryūki Matsuda

This paper consists of two notes on power series rings; the one is on the integral dependence and the other is on the henselian property. Part I is on the integral dependence and Part II is on the henselian property.

I. A note on the integral dependence in puiseux series rings.

Let $R$ be a (commutative, with $1 \neq 0$) ring and let $S$ be a (unitary) overring of $R$. Gilmer [2] is concerned with the problem of determining necessary and sufficient conditions in order that an element of $S[[X_1, \ldots, X_n]]$ be integral over $R[[X_1, \ldots, X_n]]$. We consider puiseux series rings of finite or infinite number of variables. We show in §1 that the results of Gilmer hold also in puiseux series rings of any cardinal number of variables. We denote the total quotient ring of puiseux series ring $R[[[X_i; I]]]$ by $R(((X_i; I)))$. In §2, results of §1 are applied to relate structure properties of $S(((X_i; I))/R(((X_i; I)))$ to that of $S/R$. As an appendix we state a remark on the usual (that is, of powers of natural numbers) power series rings. We show that the results of Gilmer as to a finite number of variables hold for power series rings of any infinite number of variables. This note depends deeply on Gilmer and the proofs are straight-forward generalization of his [2]. In §1 and §2, $I$ denotes any nonempty set (finite or infinite).

§1. Integral dependence in $S[[[X_i; I]]]/R[[[X_i; I]]]$.

Definition. Let $R$ be a ring and let $\{X_i; i \in I\}$ be a set of variables with the index set $I$. We set $G=\mathbb{Q}$: the rational numbers in [5] 3. Then an element $f$ of $R_{\mathbb{Q}}[[X_i; i \in I]]$ is said a puiseux series, if a common denominator of $\text{supp}_i(f)$ is bounded for every $i \in I$. The set of all puiseux series is $R[[[X_i; i \in I]]]$ (or $R[[[X_i; I]]]$). The total quotient ring of $R[[[X_i; I]]]$ is denoted by $R(((X_i; I)))$.

An element $f$ of $R[[[X_i; I]]]$ is of the form $\sum a(X_{i_1}^{e_1} \cdots X_{i_n}^{e_n})X_{i_1}^{r_1} \cdots X_{i_n}^{r_n}$ where $n$ is finite, $a(X_{i_1}^{e_1} \cdots X_{i_n}^{e_n}) \in R$, $e_1, \ldots, e_n \in \mathbb{Q}$ and $r_1, \ldots, r_n \in I$.

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\[ \text{supp}(f) = \{ e \in Q : a(X^i_1 X^i_2 \cdots X^i_n) \neq 0 \text{ for some } X^i_1 X^i_2 \cdots X^i_n \} \]

has a bounded common denominator. \( R[[[X_i; I]]] \) makes a ring canonically.

**Proposition 1.** If \( S \) is a separable field extension of a field \( R \), then in order that \( S[[[X_i; I]]] \) be integral over \( R[[[X_i; I]]] \) it is necessary that \([S: R]\) be finite.

**Proof.** If \( S/R \) is infinite-dimensional, then there exists an infinite sequence \( \{a_j\} \) of elements of \( S \) such that

\[ [R(a_1, \ldots, a_j) : R(a_1, \ldots, a_{j-1})] > j \]

for every \( j \). We set \( g = \sum_{j=0}^{\infty} a_j X^i_j \) for a fixed \( i \in I \). We have

\[ g^n + g^{n-1} f_1 + \cdots + f_n = 0, \quad f_j \in R[[[X_i; I]]]. \]

Substituting \( X_i = 0 \) \((j \neq i)\), we have

\[ g^n + g^{n-1} h_1 + \cdots + h_n = 0, \quad h_j \in R[[[X_i]]]. \]

There exists an integer \( m > 0 \) such that \( mx \in Z \) for every \( x \in \bigcup \text{supp} h_j \). Set \( X^i_1/m = Y \). The element \( g = \sum a_j Y^{m} \) of \( S[[Y]] \) is integral over \( R[[Y]] \). By \([2] 2.4\), we have

\[ [R(\text{the coefficients of } g) : R] < \infty; \]

a contradiction.

**Proposition 2.** Suppose that \( S \) is an extension field of a field \( R \) and that \( g \in S[[[X_i; I]]] \). Let \( S_1 \) be the subfield of \( S \) obtained by adjoining the coefficients of \( g \) to \( R \). If \( g \) is integral over \( R[[[X_i; I]]] \), then the separable degree \([S_1: R]_s\) is finite. Conversely if \( S_1/R \) is finite-dimensional, then \( g \) is integral over \( R[[[X_i; I]]] \).

**Proof.** Suppose that \( g \) is integral over \( R[[[X_i; I]]] \). We have

\[ g^m + g^{m-1} f_{m-1} + \cdots + g f_1 + f_0 = 0 \]

for \( f_j \in R[[[X_i; I]]] \). Denote the coefficient in \( g \) of a monomial \( X_1^{i_1} \cdots X_n^{i_n} \) by \( b(X_1^{i_1} \cdots X_n^{i_n}) \). Let \( g', f'_j \) be the power series in \( S[[[X_1, \ldots, X_n]]] \) obtained from \( g, f_j \) by the substitution \( X_i = 0 \) for \( i \in \{i_1, \ldots, i_n\} \). We have

\[ g'^m + g'^{m-1} f'_{m-1} + \cdots + g' f'_1 + f'_0 = 0. \]

Take a common denominator \( d \) of
\[ \bigcup_j \text{supp}(g') \cup \bigcup_{j,k} \text{supp}(f'_j). \]

Set \( X_1^{1/d} = Y_1 \). \( g' \in S[[Y_1, \ldots, Y_n]] \) is integral over \( R[[Y_1, \ldots, Y_n]] \). Therefore
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by [2] Th. 1.1, \( b(X_1^t, \ldots, X_n^t) \) is algebraic over \( R \). Therefore, \( S_1 \) is algebraic over \( R \). If \([S_1 : R]_s = s\) is finite, there exist \( c_1, \ldots, c_s \in S \), such that \( S_1 = Rc_1 + \cdots + Rc_s \). We have

\[
S_1[[[X_i ; I]]] = R[[[X_i ; I]]]c_1 + \cdots + R[[[X_i ; I]]]c_s.
\]

Therefore \( g \) is integral over \( R[[[X_i ; I]]] \).

**Proposition 3.** Suppose that \( S \) is a separable algebraic extension field of a field \( R \). In order that \( S[[[X_i ; I]]] \) be integral over \( R[[[X_i ; I]]] \), it is necessary and sufficient that \([S : R] < \infty\).

**Proof.** By Prop. 1 and Prop. 2.

**Proposition 4.** Suppose that \( L \) is an algebraic extension field of a field \( K \) of characteristic \( p > 0 \), purely inseparable over \( K \). Let \( L_1 \) be the subfield of \( L \) obtained by adjoining the coefficients of an element \( g \in L[[[X_i ; I]]] \) to \( K \). Then \( g \) is integral over \( K[[[X_i ; I]]] \) if and only if \( L_1 \) has finite exponent over \( K \).

**Proof.** Set \( g = \Sigma b(X_1^t, \ldots, X_n^t)X_1^{t_1} \cdots X_n^{t_n} \), \( b(X_1^t, \ldots, X_n^t) \in L \). If \( L_1 \) has finite exponent over \( K \), we have \( L_1^e \subset K \) for some \( e > 0 \). Since

\[
g^{e^*} = \Sigma b(X_1^t, \ldots, X_n^t)^{e^*}X_1^{e^*t_1} \cdots X_n^{e^*t_n} \in K[[[X_i ; I]]],
\]

\( g \) is integral over \( K[[[X_i ; I]]] \). Conversely if \( g \) is integral over \( K[[[X_i ; I]]] \), we have

\[
g^m + g^{m-1}f_{m-1} + \cdots + gf_1 + f_0 = 0
\]

for \( f_j \in K[[[X_i ; I]]] \). Take any coefficient \( b(X_1^t, \ldots, X_n^t) \) of \( g \). Let \( g', f_j \) be power series in \( L[[[X_i, \ldots, X_n]]] \) obtained by the substitution \( X_i = 0 \) for \( i \in \{i_1, \ldots, i_s\} \). We have

\[
g'^m + g'^{m-1}f_{m-1} + \cdots + g'f_1 + f_0 = 0.
\]

By [2] Th. 3.2, \( g' \) is purely inseparable over \( K((X_{i_1}, \ldots, X_{i_s})) \). Choose an integer \( t \) such that \( p^t > m \). Since \( g'^{p^t} \) belongs to \( K((X_{i_1}, \ldots, X_{i_s})) \), and \( K[[[X_i, \ldots, X_n]]] \) is integrally closed, we have \( g'^{p^t} \in K[[[X_i, \ldots, X_n]]] \) and especially \( b(X_1^t, \ldots, X_n^t)^{p^t} \in K \). Since \( L_1^t \subset K \), \( L_1 \) has finite exponent over \( K \).

**Proposition 5.** Suppose that \( L \) is a purely inseparable algebraic field extension of a field \( K \) of characteristic \( p > 0 \). Then \( L[[[X_i ; I]]] \) is integral over \( K[[[X_i ; I]]] \) if and only if \( L \) has finite exponent over \( K \).

**Proof.** By Prop. 4.

**Proposition 6.** Suppose that \( S \) is an algebraic field extension of a field
$R$ of characteristic $p > 0$. Let $S_1$ be the subfield of $S$ obtained by adjoining the coefficients of an element $g \in S[[[X_i; I]]]$ to $R$. Then $g$ is integral over $R[[[X_i; I]]]$ if and only if $[S_1 : R] = \infty$ and $S_1$ has finite exponent over $R$.

**Proof.** Suppose that $g$ is integral over $R[[[X_i; I]]]$. Let $S_0$ be the separable closure in $S_1$ over $R$. By Prop. 2, we have $[S_0 : R] = [S_1 : R] = \infty$. By Prop. 4, $S_1$ has finite exponent over $S_0$. Conversely, suppose that $[S_1 : R] = \infty$ and $S_1$ has finite exponent over $R$. By Prop. 3, $S_0[[[X_i; I]]]$ is integral over $R[[[X_i; I]]]$. By Prop. 5, $S_1[[[X_i; I]]]$ is integral over $S_0[[[X_i; I]]]$. Therefore $S_1[[[X_i; I]]]$ is integral over $R[[[X_i; I]]]$. 

**Proposition 7.** Suppose that $S$ is an algebraic field extension of a field $R$ of characteristic $p > 0$. Then $S[[[X_i; I]]]$ is integral over $R[[[X_i; I]]]$ if and only if $[S : R] = \infty$ and $S$ has finite exponent over $R$.

**Proof.** If $[S : R] = \infty$, and $S$ has finite exponent over $R$, $S[[[X_i; I]]]$ is integral over $R[[[X_i; I]]]$ by Prop. 6. Conversely, suppose that $S[[[X_i; I]]]$ is integral over $R[[[X_i; I]]]$. Let $S_0$ be the separable closure in $S$ of $R$. By Prop. 3, we have $[S_0 : R] = [S : R] = \infty$.

By Prop. 5, $S$ has finite exponent over $S_0$.

**Proposition 8.** Let $S$ be an algebraic extension field of a field $R$. If $g \in S[[[X_i; I]]]$, and if $g$ is algebraic over $R[[[X_i; I]]]$, then $g$ is integral over $R[[[X_i; I]]]$.

**Proof.** There exists a nonzero element $f$ of $R[[[X_i; I]]]$ such that $fg$ is integral over $R[[[X_i; I]]]$. Let $R_1$ be the subfield of $S$ obtained by adjoining the coefficients of $fg$ to $R$. By Prop. 2 and 6, we have $[R_1 : R] = \infty$, and $R_1$ has finite exponent over $R$. Consider any coefficient $b(X_{i_1}^e \cdots X_{i_m}^e)$ of $g$. There exists a monomial $X_{i_1}^e \cdots X_{i_m}^e$ such that the coefficient $a(X_{i_1}^e \cdots X_{i_m}^e)$ in $f$ is not zero. Let $f'$, $g'$ be power series obtained by the substitution $X_i = 0$ for $i \in \{i_1, \ldots, i_n, 1, \ldots, 1_m\}$. By [2] Lem. 5.1, the field obtained by adjoining the coefficients of $f'g'$ to $R$ is the field obtained by adjoining the coefficients of $g'$ to $R$. Therefore, we have $b(X_{i_1}^e \cdots X_{i_m}^e) \in R_1$. Therefore, $R_1$ is the field obtained by adjoining the coefficients of $g$ to $R$. By Prop. 2 and 6, $g$ is integral over $R[[[X_i; I]]]$.

**Remark.** We set $G = Q$: the rational numbers in [5] 3. We set as follows;

\[
\begin{align*}
S_{G, R}[[[X_i; I]]]_1 \cap R[[[X_i; I]]] &= R[[[X_i; I]]], \\
S_{G, R}[[[X_i; I]]]^2 \cap R[[[X_i; I]]] &= R[[[X_i; I]]]^2, \\
S_{G, R}[[[X_i; I]]]^3 \cap R[[[X_i; I]]] &= R[[[X_i; I]]]^3.
\end{align*}
\]
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We denote \( R[[\xi; I]] \) also by \( R[[\xi; 1]] \). If \( I \) is an infinite set, these 7 are mutually distinct. And these 6 but \( R[[\xi; I]] \) make rings canonically. We denote total quotient rings of these 6 by \( R(((\xi; I_1)), \ldots, R(((\xi; I_6)), R(((\xi; I_7))) \). If \( I \) is finite, say \( I = \{1, 2, \ldots, n\} \), all these 7 are identical. And we denote the ring by \( R[[\xi_1, \ldots, \xi_n]] \). The total quotient ring of \( R[[\xi_1, \ldots, \xi_n]] \) is denoted by \( R(((\xi_1, \ldots, \xi_n)) \).

We see that Prop. 1, 2, 3, 4, 5, 6, 7, 8 hold for each ring of these 6 for any nonempty set \( I \) (finite or infinite).

\[ R_0[\xi; I]^3 \cap R[[\xi; I]] = R[[\xi; I]] \]
\[ R_0[\xi; I]^3 \cap R[[\xi; I]] = R[[\xi; I]] \]
\[ R_0[\xi; I]^3 \cap R[[\xi; I]] = R[[\xi; I]] \]

\[ \text{We denote } R[[\xi; I]] \text{ also by } R[[\xi; I]]. \text{ If } I \text{ is an infinite set, these 7 are mutually distinct. And these 6 but } R[[\xi; I]] \text{ make rings canonically. We denote total quotient rings of these 6 by } R(((\xi; I_1)), R(((\xi; I_2)), R(((\xi; I_3)), R(((\xi; I_4)), R(((\xi; I_5)), R(((\xi; I))) \). If } I \text{ is finite, say } I = \{1, 2, \ldots, n\}, \text{ all these 7 are identical. And we denote the ring by } R[[\xi_1, \ldots, \xi_n]]. \text{ The total quotient ring of } R[[\xi_1, \ldots, \xi_n]] \text{ is denoted by } R(((\xi_1, \ldots, \xi_n)). \]

We see that Prop. 1, 2, 3, 4, 5, 6, 7, 8 hold for each ring of these 6 for any nonempty set \( I \) (finite or infinite).

\section{2. Structure properties of \( S(((\xi; I))/R(((\xi; I))) \).}

Let \( I \) be any nonempty set (finite or infinite) as in the previous section. Results in section 1 on the integral closure of \( R[[\xi; I]] \) in \( S[[\xi; I]] \) when \( R \) and \( S \) are fields are applied to relate structure properties of \( S(((\xi; I))/R(((\xi; I))) \) to those of \( S/R \).

**Proposition 1.** Let \( R \) be a field. Then \( R[[\xi; I]] \) is integrally closed domain.

**Proof.** If \( I \) is finite, \( R[[\xi; I]] \) is integrally closed domain. Therefore, we assume \( I \) infinite. Suppose \( 0 \neq F \in \sigma R((\xi; I)) \) is integral over \( R[[\xi; I]] \). We have \( F = f/g, f, g \in R[[\xi; I]] \). Let \( \{i_1, \ldots, i_m\} \) be a nonempty finite subset of \( I \), and \( h \in R[[\xi; I]] \). We denote the value obtained by the substitution \( \xi_i = 0, i \in \{i_1, \ldots, i_m\} \) in \( h \) by \( h(i_1, \ldots, i_m) \) in general. Let \( (I) \) be the set of \( \{i_1, \ldots, i_m\} \) such that \( f(i_1, \ldots, i_m)g(i_1, \ldots, i_m) \neq 0 \); obviously \( (I) \neq \emptyset \). We have

\[ F^n + f_1F^{n-1} + \cdots + f_{n-1}F + f_n = 0, \quad f_j \in \sigma R[[\xi; I]]. \]

Set \( f(i_1, \ldots, i_m)g(i_1, \ldots, i_m) = F(i_1, \ldots, i_m) \) for \( \{i_1, \ldots, i_m\} \in (I) \). Then we have

\[ F(i_1, \ldots, i_m)^n + f_1(i_1, \ldots, i_m)F(i_1, \ldots, i_m)^{n-1} + \cdots + f_n(i_1, \ldots, i_m) = 0. \]

The element \( F(i_1, \ldots, i_m) \) of \( R(((\xi_1, \ldots, \xi_n))) \) is integral over \( R[[\xi_1, \ldots, \xi_n]] \), hence \( F(i_1, \ldots, i_m) \in R[[\xi_1, \ldots, \xi_n]] \). Let \( i \in \{i_1, \ldots, i_m\} \in (I) \). Take a common denominator \( 0 < k \in Z \) of \( \supp_j(f) \cup \supp_j(g) \). Since

\[ f(i_1, \ldots, i_m) = F(i_1, \ldots, i_m)g(i_1, \ldots, i_m), \]

\( k \) is a common denominator of an element \( F(i_1, \ldots, i_m) \) of \( R[[\xi_1, \ldots, \xi_n]] \). Let \( \{i_1, \ldots, i_m\}, \{i_1, \ldots, i_j, i_{m+1}, \ldots, i_m\} \in (I), \{i_1, \ldots, i_j\} \subset \{i_1, \ldots, i_m\} \subset \{i_1, \ldots, i_{m+1}, \ldots, i_m\} \).
Then we have

\[ (F(i_1, \ldots, i_m, i_{m+1}, \ldots, i_n)) (i_1, \ldots, i_m) = F(i_1, \ldots, i_m). \]

Therefore, an element \( h \) of \( R[[[X_j; I]]] \) is determined canonically such that

\[ h(i_1, \ldots, i_m) = F(i_1, \ldots, i_m) \text{ for any } \{i_1, \ldots, i_m\} \subseteq (I). \]

Then we have \( F = h \in R[[[X_i; I]]] \). Therefore \( R[[[X_i; I]]] \) is integrally closed.

**Proposition 2.** Let \( R \) be a subfield of a field \( S \), let \( T \) be the algebraic closure of \( R(((X_i; I))) \) in \( S(((X_i; I))) \), and let \( J \) be the integral closure of \( R[[[X_i; I]]] \) in \( S[[[X_i; I]]] \). Then we have

\[ a) \quad T \text{ is the quotient field of } J; \text{ in fact } T = J_N, \text{ where } N = R[[[X_i; I]]] - \{0\}. \]

\[ b) \quad T = S(((X_i; I))) \text{ if and only if } J = S[[[X_i; I]]]. \]

**Proof.** \( J_N \subseteq T \) is obvious. Conversely let \( g \in T \). \( fg \) is integral over \( R[[[X_i; I]]] \) for some nonzero \( f \in R[[[X_i; I]]] \). Since \( S[[[X_i; I]]] \) is integrally closed by Prop. 1, we have \( g \in J_N \). Therefore we have \( T = J_N \). Next if \( J = S[[[X_i; I]]] \), we have \( T = S(((X_i; I))) \). Conversely if \( T = S(((X_i; I))) \), any element \( g \in S[[[X_i; I]]] \) is algebraic over \( R[[[X_i; I]]] \). By § 1, Prop. 8, \( g \) is integral over \( R[[[X_i; I]]] \). Therefore \( J = S[[[X_i; I]]] \).

**Proposition 3.** Let \( S \) be an algebraic extension field of a field \( R \), and let \( T \) be the algebraic closure of \( R(((X_i; I))) \) in \( S(((X_i; I))) \). Then we have

\[ a) \quad \text{If } S/R \text{ is separable, then } T|R(((X_i; I))) \text{ is separable.} \]

\[ b) \quad \text{If } S/R \text{ is purely inseparable, then } T|R(((X_i; I))) \text{ is purely inseparable.} \]

\[ c) \quad \text{If } S/R \text{ is normal, then } T|R(((X_i; I))) \text{ is normal.} \]

\[ d) \quad \text{If } S/R \text{ is normal of finite degree, then } T|R(((X_i; I))) \text{ is normal of finite degree.} \]

**Proof.** a): Let \( g \in T \). By Prop. 2, a), there exists \( g_1 \in S[[[X_i; I]]] \) which is integral over \( R[[[X_i; I]]] \) and \( 0 \neq f \in R[[[X_i; I]]] \) such that \( g = g_1/f \). Let \( R_1 \) be the subfield of \( S \) obtained by adjoining the coefficients of \( g_1 \) to \( R \). By § 1, Prop. 2, we have \( [R_1 : R] < \infty \). There exists a finite subset \( \{\theta_1, \ldots, \theta_s\} \subseteq R_1 \) such that \( R_1 = \Sigma R\theta_j \). We have \( R_1[[[X_i; I]]] = \Sigma R[[[X_i; I]]] \theta_j \). Therefore, \( R_1(((X_i; I))) \) is separable algebraic over \( R(((X_i; I))) \). b): We may suppose that the characteristic \( p \) is not zero. In the above proof of a), \( R_1 \) has finite exponent over \( R \) by § 1, Prop. 4. Therefore, \( R_1^m \subseteq R \) for some \( m > 0 \). Hence \( g^m \in R(((X_i; I))) \). c): Suppose first that \( S \) is separable over \( R \). Let \( R_2 \) be the normal closure of \( R_1/R \) in \( S \). Then, \( R_2(((X_i; I))) \) is normal over \( R(((X_i; I))) \). Therefore \( T \) is normal over \( R(((X_i; I))) \). Next, suppose that \( S \) is not separable over \( R \). Let \( R_0 \) be the purely inseparable closure of \( R \) in \( S \). \( S \) is separable over \( R_0 \). Let \( A \) be the algebraic closure of \( R(((X_i; I))) \) in \( R_0(((X_i; I))) \). By b), \( A \) is purely inseparable over \( R(((X_i; I))) \). Let \( B_0 \) be the algebraic closure of \( R_0(((X_i; I))) \) in \( S(((X_i; I))) \). From a)
and the first case, \( B_0 \) is normal separable over \( R_0(((X_i; I))) \). Let \( F(X) \) be the minimal polynomial of \( g \) over \( R_0(((X_i; I))) \). We have \( F(X) = \Pi(X - g^{(j)}) \) for \( g^{(j)} \in B_0 \). Since \( g \in T \), we have \( g^{(j)} \in T \). Therefore the coefficients of \( F(X) \) belong to \( A \). Hence \( F(X) \) is the minimal polynomial of \( g \) over \( A \). Therefore \( T \) is normal over \( A \). Hence \( T \) is normal over \( R(((X_i; I))) \).

d): We have \( S = Rb_1 + \cdots + Rb_s \) for \( b_j \in S \). Then \( S(((X_i; I))) = R(((X_i; I)))(b_1, \ldots, b_s) \). The assertion follows from c).

Appendix. Integral dependence in (usual) power series.

We remark in this appendix that the results of [2] hold also in (usual) power series rings of any infinite number of variables. The proofs are completely straightforward generalization of those of [2]. Throughout the appendix, \( I \) denotes an infinite set. Let \( R \) be a ring and let \( \{X_i; I\} \) be a set of variables with the index set \( I \). Then we consider the full ring \( R[[X_i; I]] \) (1) p. 66) of (usual) power series of the infinite number of variables over \( R \). We denote the total quotient ring of \( R[[X_i; I]] \) by \( R((X_i; I)) \).

**Proposition 1.** Suppose that \( S \) is an field extension of a field \( R \), and \( g \in S[[X_i; I]] \). Let \( S_1 \) be the subfield of \( S \) obtained by adjoining the coefficients of \( g \) to \( R \). Then \( g \) is integral over \( R[[X_i; I]] \) if and only if \( [S_1: R]_s < \infty \) and \( S_1 \) has finite exponent over \( R \).

**Proposition 2.** Suppose that \( S \) is an extension field of a field \( R \). Then, \( S[[X_i; I]] \) is integral over \( R[[X_i; I]] \) if and only if \( [S: R]_s < \infty \) and \( S \) has finite exponent over \( R \).

**Proposition 3.** Let \( S \) be an extension field of a field \( R \). If \( g \in S[[X_i; I]] \), and if \( g \) is algebraic over \( R[[X_i; I]] \), then \( g \) is integral over \( R[[X_i; I]] \).

**Proposition 4.** Let \( R \) be a subfield of a field \( S \), let \( T \) be the algebraic closure of \( R(X_i; I) \) in \( S((X_i; I)) \), and let \( J \) be the integral closure of \( R[[X_i; I]] \) in \( S[[X_i; I]] \).

a) \( T \) is the quotient field of \( J \); in fact \( T = J_N \), where \( N = R[[X_i; I]] - \{0\} \).

b) \( T = S((X_i; I)) \) if and only if \( J = S[[X_i; I]] \).

**Proposition 5.** Let \( S \) be an algebraic extension field of a field \( R \), and let \( T \) be the algebraic closure of \( R(X_i; I) \) in \( S((X_i; I)) \).

a) If \( S/R \) is separable, then \( T/R((X_i; I)) \) is separable.

b) If \( S/R \) is purely inseparable, then \( T/R((X_i; I)) \) is purely inseparable.

c) If \( S/R \) is normal, then \( T/R((X_i; I)) \) is normal.

Remark. As power series rings of positive integers powered and of infinite number of variables, two restricted power series rings \( R[[X_i; I]]_1, R[[X_i; I]]_2 \)
may be considered except \( R[[X_i; I]] = R[[X_i; I]]_3 ([3]) \). Propositions 1, 2, 3, 4, 5 works equally well for the two restricted power series rings also.

II. The henselian property of rings of power series.

Let \( A \) be a ring (commutative and with 1), \( p \) a maximal ideal, \( I \) any set. Let \( m_j \) be the set of power series of \( A[[X_i; I]]_j ([3]) \) the constants of which belong to \( p \) (\( j = 1, 2, 3 \)). It has been proved that, if \( A \) is henselian at \( p \), then \( A[[X_i; I]]_j \) is henselian at \( m_j \) (\( j = 1, 2, 3 \)). Let \( a \) be an ideal (not necessarily maximal) of \( A, G \) a totally ordered commutative group. In this note we extend the result as follows: If \( A \) is henselian at \( a \), then \( A_{go}[[X_i; I]]_1, A_{go}[[X_i; I]]_2, A_{go}[[X_i; I]]_3, A_{go}[[X_i; I]]_1, A_{go}[[X_i; I]]_2, A_{go}[[X_i; I]]_3 ([5] § 3) \) are all henselian at \( \mathfrak{U} \), where \( \mathfrak{U} \) is the set of power series of respective power series rings constant of which belongs to \( a \). As an appendix, we remark that puiseux series rings \( A[[X_i; I]]_1, A[[X_i; I]]_2, A[[X_i; I]]_3 \) satisfy also the henselian property.

Let \( f(Y) \in A[Y] \). We denote by \( f(Y) \) the polynomial in \( (A/a)[Y] \) the coefficients of which are obtained by the reduction mod. \( a \).

**DEFINITION 1.** A ring \( A \) is said to be henselian at an ideal \( a \), if for each monic polynomial \( f(Y) \in A[Y] \) so that \( f(Y) \) of \( (A/a)[Y] \) factors into a product \( g(Y)h(Y) \) where \( g(Y) \) and \( h(Y) \) are monic and relatively prime, then there exist monic and relatively prime polynomials \( g_0(Y) \) and \( h_0(Y) \) in \( A[Y] \) such that \( f(Y) = g_0(Y)h_0(Y), g(Y) \equiv g_0(Y)(a), h(Y) \equiv h_0(Y)(a) \).

This definition is an extension of the henselian property to any ideal (not necessarily maximal). \( A \) is always henselian at \( (0) \).

**REMARK 2.** There exist a ring \( A \) and an ideal \( a \) of \( A \) which is neither \( (0) \) nor \( (1) \) nor prime, such that \( A \) is henselian at \( a \). For example, let \( k \) be a field and set \( A = k[X]/(X^3), a = (X^2)/(X^3) \). Then \( a \) is neither \( (0) \) nor \( (1) \) nor prime, but \( A \) is henselian at \( a \).

**THEOREM 3.** Let \( a \) be an ideal of \( A, I \) any set, \( G \) a totally ordered commutative (additive) group. If \( A \) is henselian at \( a \), then \( A_{go}[[X_i; I]]_1, A_{go}[[X_i; I]]_2, A_{go}[[X_i; I]]_3, A_{go}[[X_i; I]]_1, A_{go}[[X_i; I]]_2, A_{go}[[X_i; I]]_3 \) are all henselian at \( \mathfrak{U} \), where \( \mathfrak{U} \) is the set of power series of respective rings the constant coefficient of which belongs to \( a \). If \( a \) is a maximal, then \( \mathfrak{U} \) is also a maximal ideal of respective rings.

The proof is stated below.

**COROLLARY 4.** Let \( a \) be an ideal of \( A, I \) any set. If \( A \) is henselian at \( a \), then \( A[[X_i; I]]_1, A[[X_i; I]]_2, A[[X_i; I]]_3 \) are all henselian at \( \mathfrak{U}_j \), where \( \mathfrak{U}_j \).
is the set of power series of $A[[X_i; I]]$, the constant of which belongs to $a$ ($j = 1, 2, 3$). If $a$ is maximal, then $\mathfrak{A}_j$ is also a maximal ideal of $A[[X_i; I]]$.

**Remark 5.** We may discuss the henselian property for various rings of power series of $[5] \S 3$.

**Remark 6.** Let $p$ be a maximal ideal of a ring $A$, $G = \mathbb{Z}$ with the usual order, $\mathfrak{A}$ the set of elements of $A_0[[X]]$ coefficients of which belong to $p$. Then $\mathfrak{A}$ is a maximal ideal of $A_0[[X]]$. Even if $A$ is henselian at $p$, $A_0[[X]]$ is not necessarily henselian at $\mathfrak{A}$. For example, let $k$ be a field, $T, X$ indeterminates, $A = k[[T]]$, $p = (T)$. Then, though $A$ is henselian at a maximal ideal $p$, $A_0[[X]]$ is not henselian at $\mathfrak{A}$. For, consider the following decomposition:

$$(TX^{-1} + 2) + 3Y + Y^2 \equiv (1 + Y)(2 + Y)(\mathfrak{A}).$$

**Proof of Theorem.** We prove our assertion for $A_{0a}[[X_i; I]]$. The proofs for the others are obtained by suitable modifications of the following. Set $A_{0a}[[X_i; I]] = B$. Introduce a well-order to $I$. At first if $a$ is maximal, then $\mathfrak{A}$ is a maximal ideal of $B$. Let $F(Y)$ be a monic polynomial of $B[Y]$ of degree $n$ so that $F(Y)$ factors into a product $\overline{G}'(Y)\overline{H}'(Y)$, where $\overline{G}'(Y)$ and $\overline{H}'(Y)$ are monic and relatively prime in $(B/\mathfrak{A})[Y]$. We may assume $G'(Y), H'(Y)$ are monic polynomials of $B[Y]$, and $F(Y) \neq 1$. Considering $F(Y)$ as an element of $A[Y]_{0a}[[X_i; I]]$, let $f_0(Y)$ be the constant term, $a(X_{i_1} \cdots X_{i_n})$ the coefficient of $X_{i_1} \cdots X_{i_n}$ of $F(Y)$. Considering $G'(Y)$ and $H'(Y)$ as elements of $A[Y]_{0a}[[X_i; I]]$, let $g_0(Y)$ and $h_0(Y)$ be constant terms of $G'(Y)$ and $H'(Y)$ respectively. $\overline{g}_0(Y)$ and $\overline{h}_0(Y)$ are monic and relatively prime in $(A/a)[Y]$, and we have $f_0(Y) \equiv g_0(Y)h_0(Y)(a)$. Therefore, there exist $g_0(Y)$ and $h_0(Y)$ of $A[Y]$ which are monic and relatively prime and satisfy

$$f_0(Y) = g_0(Y)h_0(Y), \quad g_0(Y) \equiv g'_0(Y)(a), \quad h_0(Y) \equiv h'_0(Y)(a).$$

Set $\deg g_0(Y) = s$, $\deg h_0(Y) = t$, $s + t = n$. At first suppose $s$ or $t$ is 0, say $s = 0$. We set $G(Y) = 1$ and $H(Y) = F(Y)$. Then $G(Y)$ and $H(Y)$ are relatively prime monic polynomials of $B[Y]$ such that

$$F(Y) = G(Y)H(Y), \quad G(Y) \equiv g'(Y)(\mathfrak{A}), \quad H(Y) \equiv h'(Y)(\mathfrak{A}).$$

Next, suppose $s$ and $t$ are greater than 0. We determine polynomials $\beta(X_{i_1} \cdots X_{i_n})$ and $\gamma(X_{i_1} \cdots X_{i_m})$ of $A[Y]$ for every monomial $X_{i_1} \cdots X_{i_n}$ of $\mathfrak{A}$. Here $\mathfrak{A}$ is the set of finite products of elements of $\{X_{i_1} \cdots X_{i_n}; a(X_{i_1} \cdots X_{i_n}) \neq 0\}$. Introduce an order to $\mathfrak{A}$ as follows. At first $1$ is the minimal element of $\mathfrak{A}$. Let $X_{i_1} \cdots X_{i_n}$, $X_{i_1} \cdots X_{i_m} \in \mathfrak{A} \setminus \{1\}$, $e_k \neq 0 \neq f_k$. If $i_1 < \cdots < i_n$, $j_1 < \cdots < j_m$. If $n < m$, set $X_{i_1} \cdots X_{i_n} < X_{j_1} \cdots X_{j_m}$. If $n = m$, $i_1 < j_1$, $i_{k+1} = j_{k+1}$, \ldots, $i_n = j_n$, then $X_{i_1} \cdots X_{i_n} \prec X_{j_1} \cdots X_{j_m}$. If $n = m$, $i_1 = j_1$, \ldots, $i_n = j_n$, $e_1 + \cdots + e_n < f_1 + \cdots + f_m$, set $X_{i_1} \cdots X_{i_n} < X_{j_1} \cdots X_{j_m}$.
If \( n = m \), for some \( k \), set Then \( \mathcal{M} \) is a well-ordered set. We define two mappings \( \beta, \gamma \) from \( \mathcal{M} \) to \( \mathcal{A}[Y] \) by the transfinite induction as follows. At first set \( \beta(1) = g_0(Y), \gamma(1) = h_0(Y) \). Let min \( (\mathcal{M} - \{1\}) = X^q_1 \cdots X^q_m \). Since \( g_0(Y) \) and \( h_0(Y) \) are relatively prime and monic of degrees \( s, t \) respectively, there exist \( \gamma(X^q_1 \cdots X^q_m), \beta(X^q_1 \cdots X^q_m) \) of \( \mathcal{A}[Y] \) of degrees less than \( t, s \) respectively such that

\[
\alpha(X^q_1 \cdots X^q_m) = g_0(Y)\gamma(X^q_1 \cdots X^q_m) + \beta(X^q_1 \cdots X^q_m)h_0(Y).
\]

Suppose \( \beta( ), \gamma( ) \) of \( \mathcal{A}[Y] \) of degrees less than \( s, t \) respectively were determined for all monomials of \( \mathcal{M} - \{1\} \) less than \( X^q_1 \cdots X^q_m \in \mathcal{M} - \{1\} \). Since \( g_0(Y) \) and \( h_0(Y) \) are monic and relatively prime of degrees \( s \) and \( t \) respectively, we find \( \beta(X^q_1 \cdots X^q_m) \) and \( \gamma(X^q_1 \cdots X^q_m) \) of \( \mathcal{A}[Y] \) of degrees less than \( s \) and \( t \) respectively such that

\[
\alpha(X^q_1 \cdots X^q_m) = \sum_{e_j + e_j' = e_j} \beta(X^q_1 \cdots X^q_m)\gamma(X^q_1 \cdots X^q_m).
\]

The elements \( \Sigma\beta(X^q_1 \cdots X^q_m)X^q_1 \cdots X^q_m \) and \( \Sigma\gamma(X^q_1 \cdots X^q_m)X^q_1 \cdots X^q_m \) of \( \mathcal{A}[Y]_{G_0[[X_1; I]]} \) determine canonically elements \( G(Y) \) and \( H(Y) \) of \( B \) respectively. Set

\[
(B) = \{e_1 + \cdots + e_n; \alpha(X^q_1 \cdots X^q_m) \neq 0, e_1 + \cdots + e_n \neq 0\}.
\]

We confer \([4]\). Let \( V((B)) \) be the set of all finite sequences of elements of \( (B) \). Introduce the majoring order to \( V((B)) \). The mapping \( \sigma: (\alpha_1, \ldots, \alpha_n) \mapsto \alpha_1 + \cdots + \alpha_n \) from \( V((B)) \) to \( G_0 \) is a strict homomorphism.

(i) If \( F(Y) \in (A_{G_0}[[X_1; I]])[Y] \), then \( G(Y), H(Y) \in (A_{G_0}[[X_1; I]])[Y] \).

(ii) Suppose \( F(Y) \in (A_{G_0}[[X_1; I]])[Y] \). Then \( V((B)) \) has the finite basis property. Therefore, \( \sigma V((B)) \) is well-ordered and \( \sigma^{-1}(\alpha) \) is a finite set for each \( \alpha \in G_0 \). Hence \( G(Y), H(Y) \in (A_{G_0}[[X_1; I]])[Y] \).

(iii) Suppose \( F(Y) \in (A_{G_0}[[X_1; I]])[Y] \). Taking into account (ii), we see \( G(Y), H(Y) \in (A_{G_0}[[X_1; I]])[Y] \).

(iv) If \( F(Y) \in (A_{G_0}[[X_1; I]])[Y] \), then \( G(Y), H(Y) \in (A_{G_0}[[X_1; I]])[Y] \).

(v) Suppose \( F(Y) \in (A_{G_0}[[X_1; I]])[Y] \). Then \( V((B)) \) has the finite basis property. Therefore \( \sigma V((B)) \) is a well-ordered subset of \( G_0 \). Hence \( G(Y), H(Y) \in (A_{G_0}[[X_1; I]])[Y] \).

From the way of construction, \( G(Y) \) and \( H(Y) \) are monic polynomials such that

\[
F(Y) = G(Y)H(Y), \quad G(Y) = \gamma'((Y)(\mathfrak{F}), \quad H(Y) = G'((Y)(\mathfrak{F})).
\]

If we show that \( G(Y) \) and \( H(Y) \) are relatively prime in \( B[Y] \), the proof is complete. Again we determine polynomials \( \beta'(X^q_1 \cdots X^q_m) \) and \( \gamma'(X^q_1 \cdots X^q_m) \) of \( \mathcal{A}[Y] \) for
On the Integral Dependence, the Henselian Property in Power Series Rings

Every monomials of \( \mathfrak{M} \) also by the transfinite induction as follows. There exist polynomials \( \beta'(1) \) and \( \gamma'(1) \) of degrees less than \( t \) and \( s \) respectively such that

\[
g_0(Y)\beta'(1) + h_0(Y)\gamma'(1) = 1.
\]

Suppose we determined polynomials \( \beta'(\ ) \) and \( \gamma'(\ ) \) of \( A[Y] \) of degrees less than \( t \) and \( s \) respectively for all monomials of \( \mathfrak{M} \) less than \( X_{t_1}^{e_1} \cdots X_{t_n}^{e_n} \). There exist polynomials \( \beta'(X_{t_1}^{e_1} \cdots X_{t_n}^{e_n}) \) and \( \gamma'(X_{t_1}^{e_1} \cdots X_{t_n}^{e_n}) \) of \( A[Y] \) of degrees less than \( t \) and \( s \) respectively, such that

\[
\sum_{m, e_j + e_j = e_j} \{ \beta(X_{t_1}^{e_1} \cdots X_{t_n}^{e_n}) \beta'(X_{t_1}^{e_1} \cdots X_{t_n}^{e_n}) + \gamma(X_{t_1}^{e_1} \cdots X_{t_n}^{e_n}) \gamma'(X_{t_1}^{e_1} \cdots X_{t_n}^{e_n}) \} = 0.
\]

Elements \( \Sigma \beta'(X_{t_1}^{e_1} \cdots X_{t_n}^{e_n})X_{t_1}^{e_1} \cdots X_{t_n}^{e_n} \) and \( \Sigma \gamma'(X_{t_1}^{e_1} \cdots X_{t_n}^{e_n})X_{t_1}^{e_1} \cdots X_{t_n}^{e_n} \) of \( A[Y]_{Go[\Xi; I]}[X_i; I] \) determine the elements \( G''(Y) \) and \( H''(Y) \) of \( B[Y] \) canonically. Then we have

(i') If \( F(Y) \in (A_{Go[\Xi; I]}[X_i; I], \{Y\})[Y] \), then \( G''(Y), H''(Y) \in (A_{Go[\Xi; I]}[X_i; I], \{Y\})[Y] \).

(ii') If \( F(Y) \in (A_{Go[\Xi; I]}[X_i; I], \{Y\})[Y] \), then \( G''(Y), H''(Y) \in (A_{Go[\Xi; I]}[X_i; I], \{Y\})[Y] \).

(iii') If \( F(Y) \in (A_{Go[\Xi; I]}[X_i; I], \{Y\})[Y] \), then \( G''(Y), H''(Y) \in (A_{Go[\Xi; I]}[X_i; I], \{Y\})[Y] \).

(iv') If \( F(Y) \in (A_{Go[\Xi; I]}[X_i; I], \{Y\})[Y] \), then \( G''(Y), H''(Y) \in (A_{Go[\Xi; I]}[X_i; I], \{Y\})[Y] \).

(v') If \( F(Y) \in (A_{Go[\Xi; I]}[X_i; I], \{Y\})[Y] \), then \( G''(Y), H''(Y) \in (A_{Go[\Xi; I]}[X_i; I], \{Y\})[Y] \).

(vi') If \( F(Y) \in (A_{Go[\Xi; I]}[X_i; I], \{Y\})[Y] \), then \( G''(Y), H''(Y) \in (A_{Go[\Xi; I]}[X_i; I], \{Y\})[Y] \).

By the construction, we have

\[
\]

That is to say, \( G(Y) \) and \( H(Y) \) are relatively prime.

Appendix. The henselian property of puiseux series rings.

Let \( A \) be a ring, \( I \) any set. As an appendix, we remark that puiseux series rings have the henselian property also.

**Theorem.** Let \( a \) be an ideal of a ring \( A \), \( I \) any set. If \( A \) is henselian at \( a \), then \( A[[X_i; I]], A[[X_i; I]], A[[X_i; I]], A[[X_i; I]], A[[X_i; I]], A[[X_i; I]] \) are all henselian at \( \mathfrak{W}' \), where \( \mathfrak{W}' \) is the set of puiseux series of respective rings the constant coefficients of which belong to \( a \). If \( a \) is maximal, then \( \mathfrak{W}' \) is also a maximal ideal of the respective rings.

**Proof.** Let \( B' \) be one of the above 6 rings. At first if \( a \) is maximal, then \( \mathfrak{W}' \) is a maximal ideal of \( B' \). Let \( F(Y) \) be a monic polynomial of \( B'[Y] \) of degree \( n \) so that \( F(Y) = \overline{G(Y)} \overline{H(Y)} \), where \( \overline{G(Y)} \) and \( \overline{H(Y)} \) are monic and relatively
prime in \((B'/\mathfrak{M})[Y]\). We may assume that \(G'(Y), H'(Y)\) are monic polynomials of \(B'[Y]\). Set \(G = \mathcal{Q}\): the rational numbers with the usual order, and \(A_{\mathcal{Q}}[X_i; I] = B\). Let \(\mathfrak{M}\) be the set of elements of \(B\) constants of which belong to \(a\). We apply the proof of Theorem 3. By the definition of \(\mathfrak{M}\), we have \(G(Y), H(Y) \in (A[[X_i; I]])[Y]\). From (i), (ii),..., (vi), we have \(G(Y), H(Y) \in B'[Y]\). Also we have

\[ F(Y) = G(Y)H(Y), \quad G(Y) \equiv G'(Y)(\mathfrak{M}'), \quad H(Y) \equiv H'(Y)(\mathfrak{M}'). \]

Again by the definition of \(\mathfrak{M}\), we have \(G''(Y), H''(Y) \in (A[[X_i; I]])[Y]\). From (i'), (ii'),..., (vi'), we have \(G''(Y), H''(Y) \in B'[Y]\). Also we have

\[ G(Y)G''(Y) + H(Y)H''(Y) = 1. \]

That is to say, \(G(Y)\) and \(H(Y)\) are relatively prime.

References