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Some Oscillation and Asymptotic Properties for Linear Differential Equations

V. A. STAIKOS* and Ch. G. PHILOS*

In this paper we consider the n-th order \( n > 1 \) general ordinary differential equation

\[
(r_{n-1}(t)(r_{n-2}(t)\cdots(r_1(t)(r_0(t)x(t)))')')' + a(t)x(t) = b(t), \quad t \geq t_0
\]

where the functions \( r_i (i=0, 1, \ldots, n-1) \) are positive at least on the interval \([t_0, \infty)\). The continuity of the functions \( a, b \) and \( r_i (i=0, 1, 2, \ldots, n-1) \) as well as sufficient smoothness to guarantee the existence of solutions of (E) on an infinite subinterval of \([t_0, \infty)\) will be assumed without mention. In what follows the term "solution" is always used only for such solutions \( x(t) \) of (E) which are defined for all large \( t \). The oscillatory character is considered in the usual sense, i.e. a continuous real-valued function which is defined on an interval of the form \([T, \infty)\) is called oscillatory if it has no last zero, and otherwise it is called nonoscillatory.

We give here some conditions to ensure that

\[
\lim_{t \to \infty} x(t) = 0
\]

for all oscillatory solutions of the equation (E). However for \( b = 0 \), the same conditions guarantee that all eventually nontrivial solutions of the differential equation

\[
(r_{n-1}(t)(r_{n-2}(t)\cdots(r_1(t)(r_0(t)x(t)))')')' + a(t)x(t) = 0
\]

are nonoscillatory. The technique used is an adaptation of that of Singh [3] which concerns the particular case \( r_0=1, r_1=r \) and \( r_2=\cdots=r_{n-1}=1 \).

**Theorem 1.** Consider the differential equation (E) subject to the conditions:

\[
(R_0) \quad \lim_{t \to \infty} r_0(t) > 0,
\]

\[
(A) \quad \int_{s_n-1}^{\infty} \frac{1}{r_1(s)} \int_{s_1}^{s_2} \frac{1}{r_2(s)} \cdots \int_{s_{n-2}}^{s_{n-1}} \frac{1}{r_{n-1}(s_{n-1})} |a(s)| ds ds_{n-1} \cdots ds_2 ds_1 < \infty
\]

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Then for every oscillatory solution \( x \) of the differential equation (E),
\[
\lim_{t \to \infty} x(t) = 0.
\]

**Proof.** Let \( x \) be an oscillatory solution of the differential equation (E). Without loss of generality we suppose that \( x \) is a solution of (E) on the whole interval \([t_0, \infty)\).

Consider the functions \( D_r^{(k)} x \) \((k = 0, 1, \ldots, n-1)\) which are defined on the interval \([t_0, \infty)\) as follows:
\[
D_r^{(0)} x = r_0 x
\]
and
\[
D_r^{(k)} x = r_k (D_r^{(k-1)} x) \quad (k = 1, 2, \ldots, n-1).
\]

Now, we assume that
\[
\limsup_{t \to \infty} |(D_r^{(0)} x)(t)| > d
\]
for some \( d > 0 \). Moreover, because of condition \((R_0)\), there exists a constant \( c > 0 \) with
\[
r_0(t) \geq c \quad \text{for every } t \geq t_0.
\]

Thus, by conditions \((A)\) and \((B)\), we have that for some \( T \geq t_0,\)
\[
\int_{T}^{\infty} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \frac{1}{r_2(s_2)} \cdots \int_{s_{n-2}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})}
\]
\[
\int_{s_{n-1}}^{\infty} |a(s)| ds \ ds_{n-1} \cdots ds_{2} ds_{1} \leq \frac{c}{2}
\]
and
\[
\int_{T}^{\infty} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \frac{1}{r_2(s_2)} \cdots \int_{s_{n-2}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})}
\]
\[
\int_{s_{n-1}}^{\infty} |b(s)| ds \ ds_{n-1} \cdots ds_{2} ds_{1} \leq \frac{d}{2}.
\]

Since the solution \( x \) is oscillatory, the same holds for the functions \( D_r^{(k)} x \) \((k = 0, 1, \ldots, n-1)\) and consequently we can choose \( \tau_1 > \tau_2 > \cdots > \tau_{n-1} > t_1 > T \)
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with

\[(D^{(0)}_t x)(t_1) = 0\]

and

\[(D^{(k)}_t x)(t_k) = 0 \quad (k = 1, 2, \ldots, n - 1).\]

Furthermore, we consider, by (1), a \(T_0 > \tau_1\) with

\[|(D^{(0)}_t x)(T_0)| > d\]

and next a \(t_2 > T_0\) with

\[(D^{(0)}_t x)(t_2) = 0.\]

Now, on repeated integration from equation (E) we have

\[\pm (D^{(0)}_t x)(t) + \int_{t_1}^{t} \frac{1}{r_1(s_1)} \int_{s_1}^{t_2} \frac{1}{r_2(s_2)} \cdots \int_{s_{n-2}}^{t_{n-1}} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{t_{n-1}} a(s)x(s)ds \, ds_{n-1} \cdots ds_2 ds_1 = \]

\[= \int_{t_1}^{t_2} \frac{1}{r_1(s_1)} \int_{s_1}^{t_1} \frac{1}{r_2(s_2)} \cdots \int_{s_{n-2}}^{t_{n-1}} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{t_{n-1}} b(s)ds \, ds_{n-1} \cdots ds_2 ds_1\]

for every \(t \in [t_1, t_2]\) and consequently, since \(t_2 > \tau_1 > \tau_2 > \cdots > \tau_{n-1} > t_1 > T\), we get

\[|(D^{(0)}_t x)(t)| \leq \int_{t_1}^{t_2} \frac{1}{r_1(s_1)} \int_{s_1}^{t_2} \frac{1}{r_2(s_2)} \cdots \int_{s_{n-2}}^{t_{n-1}} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{t_{n-1}} |a(s)| \, ds \, ds_{n-1} \cdots ds_2 ds_1 \]

\[+ \int_{t_1}^{t_2} \frac{1}{r_1(s_1)} \int_{s_1}^{t_2} \frac{1}{r_2(s_2)} \cdots \int_{s_{n-2}}^{t_{n-1}} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{t_{n-1}} |b(s)| \, ds \, ds_{n-1} \cdots ds_2 ds_1.\]

This immediately gives

\[1 \leq \frac{M}{M^*} \int_{t_1}^{t_2} \frac{1}{r_1(s_1)} \int_{s_1}^{t_2} \frac{1}{r_2(s_2)} \cdots \int_{s_{n-2}}^{t_{n-1}} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{t_{n-1}} |a(s)| \, ds \, ds_{n-1} \cdots ds_2 ds_1 \]

\[+ \frac{1}{M^*} \int_{t_1}^{t_2} \frac{1}{r_1(s_1)} \int_{s_1}^{t_2} \frac{1}{r_2(s_2)} \cdots \int_{s_{n-2}}^{t_{n-1}} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{t_{n-1}} |b(s)| \, ds \, ds_{n-1} \cdots ds_2 ds_1,\]
Thus, by virtue of (3) and (4), we obtain

\[ 1 \leq \frac{M}{M^*} \cdot \frac{c}{2} + \frac{d}{2M^*}. \]

But, by (5) and (2), we have

\[ M^* > d \quad \text{and} \quad M^* \geq c \cdot M \]

and consequently the contradiction

\[ 1 \leq \frac{M}{M^*} \cdot \frac{c}{2} + \frac{d}{2M^*} < \frac{1}{2} + \frac{1}{2} = 1. \]

We have just proved that (1) fails and hence

\[ \lim_{t \to \infty} |(D_r^{(0)}x)(t)| = 0 \]

which, by condition (R0), gives

\[ \lim_{t \to \infty} x(t) = 0. \]

**Theorem 2.** Consider the differential equation \( (E)_0 \) subject to the conditions (R0) and (A).

Then every eventually nontrivial solution of the differential equation \( (E)_0 \) is nonoscillatory.

**Proof.** Let \( x \) be an eventually nontrivial oscillatory solution of \( (E)_0 \) on the whole interval \([t_0, \infty)\). As in the proof of Theorem 1, we can consider \( c \) and \( T \) satisfying (2) and (3). Similarly, we can choose again \( \tau_1 > \tau_2 > \cdots > \tau_{n-1} > t_1 > T \) in exactly the same way. Next, since \( x \) is eventually nontrivial and oscillatory, \( T_0 \) and \( t_2 \) can be chosen so that \( t_2 > T_0 > \tau_1 \) and

\[ |(D_r^{(0)}x)(T_0)| > 0 \quad \text{and} \quad (D_r^{(0)}x)(t_2) = 0. \]

As in the proof of Theorem 1, we obtain

\[ |(D_r^{(0)}x)(t)| \leq \int_{t_1}^{t_2} \frac{1}{r_1(s_1)} \int_{s_1}^{t_2} \frac{1}{r_2(s_2)} \cdots \int_{s_{n-1}}^{t_2} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{t_2} |a(s)| |x(s)| ds \, ds_{n-1} \cdots ds_2 \, ds_1 \]

for every \( t \in [t_1, t_2] \) and hence

\[ M^* \leq M \int_{t_1}^{t_2} \frac{1}{r_1(s_1)} \int_{s_1}^{t_2} \frac{1}{r_2(s_2)} \cdots \int_{s_{n-1}}^{t_2} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{t_2} |a(s)| \, ds \, ds_{n-1} \cdots ds_2 \, ds_1, \]
where $M = \max_{t \in [t_1, t_2]} |x(t)|$ and $M^* = \max_{t \in [t_1, t_2]} |(D_r(t)x)(t)|$.

Therefore, by (2) and (3), we have

$$Mc \leq M^* \leq M^*_c\frac{C}{2}$$

and consequently $M^* = 0$, which is a contradiction since by the definition of $T_0$,

$$M^* \geq |(D_r(t)x)(T_0)| > 0.$$

We shall now clarify the importance of Theorems 1 and 2 by applying them in the particular case where for some integer $m$, $1 \leq m \leq n-1$, we have $r_j = 1$ for $j \neq n-m$ and $r_{n-m} = r$.

More precisely, we give two corollaries concerning the differential equation

$$(E_m)$$

**COROLLARY 1.** Consider the differential equation $(E_m)$ subject to the conditions:

$$(A_m)$$

$$\int_{\infty}^{\infty} \frac{t^{n-m-1}}{r(t)} \int_{t}^{\infty} (s-t)^{m-1} |a(s)| \, ds \, dt < \infty$$

and

$$(B_m)$$

$$\int_{\infty}^{\infty} \frac{t^{n-m-1}}{r(t)} \int_{t}^{\infty} (s-t)^{m-1} |b(s)| \, ds \, dt < \infty.$$

Then for every oscillatory solution $x$ of the differential equation $(E_m)$,

$$\lim_{t \to \infty} x(t) = 0.$$

**PROOF.** We have the formula

$$\int_{\infty}^{\infty} \int_{\infty}^{\infty} (s-u)^k p(s) \, ds \, dv = \int_{\infty}^{\infty} \int_{u}^{\infty} \frac{(s-u)^{k+1}}{k+1} p(s) \, ds,$$

where $p$ is a continuous nonnegative function on $[u, \infty)$ and $k$ a nonnegative integer. By this formula, it is a matter of elementary calculus to see that in the considered case the conditions (A) and (B) follow from $(A_m)$ and $(B_m)$ respectively.

**COROLLARY 2.** Consider the differential equation $(E_m)_0$.

$$(E_m)_0$$

$$[r(t)x^{(n-m)}(t)]^{(m)} + a(t)x(t) = 0,$$

subject to the condition $(A_m)$.

Then every eventually nontrivial solution of the differential equation $(E_m)_0$ is nonoscillatory.
Remark. The above corollaries 1 and 2 are generalizations of the results in [3] still in the particular case \( m = n - 1 \). ([1], [2])

References

