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引用
Bulletin of the Faculty of Science, Ibaraki University. Series A, Mathematics, 7: 13-16

公開年
1975

URL
http://hdl.handle.net/10109/2865

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Correspondence of Boundaries of Open Sets in Coverings

Nozomu Mochizuki

1. Let \( \pi: \tilde{X} \to X \) be a covering map and \( \bar{D} \) be an open subset of \( \tilde{X} \). The boundary of \( \bar{D} \) will be denoted by \( \partial \bar{D} \). Our main purpose is to discuss a relation between the condition \( \pi(\partial \bar{D}) \subseteq \partial \pi(\bar{D}) \) and a property of \( \pi|\bar{D} \), the restriction of \( \pi \) to \( \bar{D} \). Arguments used are quite elementary.

2. In what follows, all spaces are assumed to be locally arcwise connected Hausdorff spaces. If \( \tilde{X} \) is such a space, every open connected subset is arcwise connected; every component of an open subset is open. A continuous map \( \tilde{\sigma}: [0, 1] \to \tilde{X} \) with \( \tilde{\sigma}(0)=\tilde{p} \), \( \tilde{\sigma}(1)=\tilde{q} \) will simply be called an arc \( \tilde{\sigma} \) in \( \tilde{X} \) with the initial point \( \tilde{p} \) and the terminal point \( \tilde{q} \); for this, we shall also say that \( \tilde{\sigma} \) joins \( \tilde{p} \) and \( \tilde{q} \).

**Lemma 1.** Let \( \pi: \tilde{X} \to X \) be a continuous map of \( \tilde{X} \) into \( X \) and let \( D, D' \) be open subsets of \( \tilde{X}, X \), respectively, such that \( \pi(D) \subseteq D \). Then, \( \pi(\partial D) \subseteq \partial D \) if and only if every component of \( D \) is also a component of \( \pi^{-1}(D) \).

**Proof.** Let \( \pi(\partial D) \subseteq \partial D \). Let \( \bar{D} = \bigcup_{i \in I} \bar{D}_i \) be the decomposition of \( \bar{D} \) into components where \( I \) denotes an index set. Since \( \partial \bar{D}_i \subseteq \partial \bar{D} \), we have \( \pi(\partial \bar{D}_i) \subseteq \partial \bar{D} \) for every \( i \in I \). The open subset \( \pi^{-1}(D) \) is decomposed into components such as \( \pi^{-1}(D) = \bigcup_{j \in J} \bar{G}_j \), in which every \( \bar{D}_i \) is contained in some \( \bar{G}_j \). We suppose that \( \bar{D}_i \subseteq \bar{G}_j \) for some \( i \in I, j \in J \). Let \( \tilde{p} \in \bar{D}_i, \tilde{q} \in \bar{G}_j - \bar{D}_i \). There exists an arc \( \tilde{\sigma} \) in \( \bar{G}_j \) which joins \( \tilde{p} \) and \( \tilde{q} \). Since \( \tilde{\sigma} \) is connected, we have \( \tilde{\sigma} \cap \partial \bar{D}_i \neq \emptyset \). It follows that \( D \cap \pi(\partial \bar{D}_i) \neq \emptyset \); this is a contradiction.

If, conversely, every component \( \bar{D}_i \) is a component of \( \pi^{-1}(D) \), we have \( \pi^{-1}(D) = \bar{D} \cup \bar{G} \) where \( \bar{G} \) is an open subset and \( \bar{D} \cap \bar{G} = \emptyset \). Suppose that \( \pi(\tilde{p}) \in \partial \bar{D} \) for a point \( \tilde{p} \in \partial \bar{D} \), then \( \tilde{p} \in \bar{D} \). From this follows that \( \tilde{p} \in \bar{G} \), which implies that \( \bar{D} \cap \bar{G} \neq \emptyset \). This completes the proof.

3. In this section, \( \pi: \tilde{X} \to X \) denotes a continuous map of \( \tilde{X} \) onto \( X \). For an open subset \( \bar{D} \) of \( \tilde{X} \) we shall put \( \pi_\bar{D} = \pi|\bar{D} \), the restriction of \( \pi \).

**Theorem 1.** (1) Let \( \pi: \tilde{X} \to X \) be a local homeomorphism, \( \bar{D} \) be an open subset of \( \tilde{X} \). If \( \pi_\bar{D}: \bar{D} \to D \) is a covering map, then \( \pi(\partial \bar{D}) \subseteq \partial D \).

Received February 7, 1974.

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(2) Let $\pi: \tilde{X} \to X$ be a covering map and let $\tilde{D}, D$ be open subsets of $\tilde{X}, X$ respectively such that $\pi(\tilde{D}) \subset D$. If $\pi(\partial \tilde{D}) \subset \partial D$, then $\pi_{\partial}: \tilde{D} \to \pi(\tilde{D})$ is a covering map and $\pi(\partial \tilde{D})$ lies dense in $\partial \pi(\tilde{D})$.

**Proof.** (1) Suppose that $\pi_{\partial}: \tilde{D} \to D$ is a covering map. We must show that each component of $\tilde{D}$ is a component of $\pi^{-1}(D)$. Let $\tilde{D}_0$ be a component of $\tilde{D}$, then $\tilde{D}_0 \subset \mathcal{G}_0$ for a component $\mathcal{G}_0$ of $\pi^{-1}(D)$. We fix a point $\tilde{p} \in \tilde{D}_0$. For an arbitrary point $\tilde{q} \in \mathcal{G}_0$, an arc $\tilde{\sigma}$ in $\mathcal{G}_0$ joins $\tilde{p}$ and $\tilde{q}$, and $\pi\tilde{\sigma}$ is an arc in $D$. The covering map $\pi_{\partial}: \tilde{D} \to D$ lifts $\pi\tilde{\sigma}$ to an arc $\pi\tilde{\tau}$ in $\tilde{D}$ with the initial point $\tilde{p}$. Since $\tilde{D}_0$ is a component of $\tilde{D}$, $\pi\tilde{\tau}$ is in $\tilde{D}_0$. We have $\pi\tilde{\sigma} = \pi\tilde{\tau}$ and since, as is well known, the lifting of an arc by a local homeomorphism is unique, it follows that $\tilde{\sigma} = \tilde{\tau}$, so $\tilde{q} \in \tilde{D}_0$. Thus we have $\tilde{D}_0 = \mathcal{G}_0$.

(2) Let $\tilde{D} = \bigcup_{i \in I} \tilde{D}_i$ and $D = \bigcup_{a \in A} D_a$ be decompositions into components. Let $B = \{a \in A | (D_a \cap \pi^{-1}(D_a) \neq \emptyset)\}$. Then, each $\tilde{D}_i$ is contained in some $\pi^{-1}(D_a), a \in B$. Let $I_a = \{i \in I | (\tilde{D}_i \subset \pi^{-1}(D_a)\}, a \in B$, and let $\tilde{D}_a = \bigcup_{i \in I_a} \tilde{D}_i$. Every $\tilde{D}_a, i \in I_a$, is a component of $\pi^{-1}(D)$ from the assumption, hence a fortiori a component of $\pi^{-1}(D_a)$. From this fact follows as usual that $\pi(\tilde{D}_a) = D_a, i \in I_a$, and $\pi | \tilde{D}_a: \tilde{D}_a \to D_a$ is a covering map. It is clear that $\tilde{D} = \bigcup_{a \in B} \tilde{D}_a, \pi(\tilde{D}) = \bigcup_{a \in B} D_a$, therefore, $\pi_{\partial}: \tilde{D} \to \pi(\tilde{D})$ is a covering map. Now, let $D_0 = \pi(\tilde{D})$. Then, $\pi(\partial \tilde{D}) \subset \partial D_0$ from the above. To see that $\pi(\partial \tilde{D})$ is dense in $\partial D_0$, we suppose on the contrary that there exists $p \in \partial D_0$ such that $p \notin \pi(\partial \tilde{D})$. Let $U$ be an open connected neighborhood of $p$ such that $\pi(\partial \tilde{D}) \cap U = \emptyset$. An arc $\sigma$ in $U$ joins $p$ and a point $q \in D_0 \cap U$. There exist $\tilde{q} \in \tilde{D}, \pi(\tilde{q}) = q$, and the lifting $\tilde{\sigma}$ of $\sigma$ by $\pi$ with the initial point $\tilde{q}$ and the terminal point $\tilde{p}$. From $p \in \partial D_0$ follows that $p \notin \tilde{D};$ hence, $\tilde{\sigma} \cap \partial \tilde{D} = \emptyset$. This implies that $\pi(\partial \tilde{D}) \cap U \neq \emptyset$, a contradiction. The proof is completed.

**Corollary 1.** Let $\pi: \tilde{X} \to X$ be a covering map, $\tilde{D} \subset \tilde{X}$ be an open subset and $D = \pi(\tilde{D})$. Then $\pi_{\partial}: \tilde{D} \to D$ is a covering map if and only if $\pi(\partial \tilde{D}) \subset \partial D$.

The following Lemma 2 is essentially due to Corollary 1 to Theorem 7 in [1]; arguments there, combined with our Corollary 1, yield Corollary 2 below.

**Lemma 2.** Let $\pi: \tilde{X} \to X$ be a local homeomorphism and $D = \pi(\tilde{D})$ where $\tilde{D}$ is an open subset of $\tilde{X}$. If $\pi_{\partial}: \tilde{D} \to D$ is a closed map, then $\pi(\partial \tilde{D}) \subset \partial D$.

**Proof.** Let $\partial \tilde{D}, \partial = \pi(\partial \tilde{D})$. Choose open neighborhoods $\tilde{U}, U$ of $\partial, p$ respectively for which $\pi | \tilde{U}: \tilde{U} \to U$ is a homeomorphism. There exists a net $\{\tilde{p}_i\} \subset \tilde{D} \cap \tilde{U}$ such that $\tilde{p}_i \to \partial$. The subset $\pi(\tilde{D} \cap \{\tilde{p}_i\})$ is closed in $D$ by assumption, and $\pi(\tilde{p}_i) \to p$. Let $p \in D$, then $p$ belongs to this subset; hence $p = \pi(\tilde{q})$ for some $\tilde{q} \in \tilde{D} \cap \{\tilde{p}_i\}$. If $\tilde{q} \notin \tilde{U}$, we have $\partial = \tilde{q}$, a contradiction. On the other hand, if $\tilde{q} \in \tilde{U}$, we have $p \in \partial U$, since $\tilde{q} \in \partial \tilde{U}$ and $\pi(\partial \tilde{U}) \subset \partial U$; this is also a contradiction. The proof is completed.
COROLLARY 2. Let $\mathcal{B}$ be a relatively compact open subset of $\mathcal{X}$ in Corollary 1, then the following are equivalent each other.

1. $\pi_\mathcal{B}: \mathcal{B} \rightarrow D$ is a covering map.
2. $\pi_\mathcal{B}: \mathcal{B} \rightarrow D$ is a closed map.
3. $\pi(\partial\mathcal{B}) \subset \partial D$.
4. $\pi(\partial\mathcal{B}) = \partial D$.

The relation $\pi(\partial\mathcal{B}) = \partial D$ does not hold in general in Corollary 1. Let $\mathcal{X}_n = \{(x, 0) : 0 < x < 3\}$, $\mathcal{X} = \bigcup_{n=1}^{\infty} \mathcal{X}_n$, and $X = \{(x, 0) : 0 < x < 3\}$. By $\pi(x, y) = (x, 0)$, $\pi: \mathcal{X} \rightarrow X$ is a covering map. Let $a_n = 2(1 - \frac{1}{2^n})$ and define $\mathcal{B}_n = \{(x, n) : 0 < x < a_n\}$, $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$. Then, $D = \pi(\mathcal{B}) = \{(x, 0) : 0 < x < 2, x \neq a_n, n = 1, 2, \ldots\}$ and $\pi_\mathcal{B}: \mathcal{B} \rightarrow D$ is a homeomorphism. On the other hand, $\pi(\partial\mathcal{B}) = \bigcup_{n=1}^{\infty} \{(a_n, 0)\}$ and $\partial D = \pi(\partial\mathcal{B}) \cup \{(2, 0)\}$.

4. In this section, we shall introduce a concept of connectedness of an open set near a boundary point. Let $D$ be an open subset of $X$ and let $p \in \partial D$. We shall say that $D$ is connected near $p$ if the following is satisfied: For any neighborhood $U$ of $p$, there exists an open neighborhood $V$ of $p$ such that $V \subset U$ and $D \cap V$ is connected. If $D$ is connected near every boundary point, $D$ will be said to be connected near the boundary.

Let $D = \bigcup_{a \in A} D_a$ be the decomposition into components; let $p \in \partial D$. Then, $D$ is connected near $p$ if and only if there exists $\alpha \in A$ such that $p \in \partial D_\alpha$ where $D_\alpha$ is connected near $p$ and $p \in \bigcup_{\beta \neq \alpha} D_\beta$. In particular, if $D$ is connected near the boundary, we have $\partial D = \bigcup_{a \in A} \partial D_a$, and every $D_a$ is connected near the boundary.

Lemma 3. Let $\pi: \mathcal{X} \rightarrow X$ be a local homeomorphism and $\pi_\mathcal{B}: \mathcal{B} \rightarrow D$ be a covering map for an open subset $\mathcal{B} \subset \mathcal{X}$. Let $p \in \partial \mathcal{B}$ and $p = \pi(\bar{p})$. If $D$ is connected near $p$, then $\bar{p}$ is connected near $\bar{p}$.

Proof. Let $\bar{W}$ be a neighborhood of $\bar{p}$. We can choose open neighborhoods $\bar{U} \subset \bar{W}$, $U$ of $\bar{p}$, $p$ respectively such that $\pi(\bar{U}) : \bar{U} \rightarrow U$ is a homeomorphism and $D \cap U$ is connected. Clearly, we have $\pi(\partial \bar{U}) \subset \partial U$. We shall see that $\pi|\partial \bar{B} : \partial \bar{B} \cap \bar{U} \rightarrow D \cap U$ is a homeomorphism; for this, it is enough to see that $\pi|\partial \bar{B} \cap \bar{U}$ is a map onto $D \cap U$. Let $\bar{q}_0 \in \partial \bar{B} \cap \bar{U}$. Clearly, $\pi(\bar{q}_0) \in D \cap U$. For an arbitrary point $q \in D \cap U$, an arc $\sigma$ in $D \cap U$ joins $\pi(\bar{q}_0)$ and $q$; this is lifted by $\pi_\mathcal{B}$ to an arc $\bar{\sigma}$ in $\mathcal{B}$ with the initial point $\bar{q}_0$ and the terminal point $\bar{q} \in \mathcal{B}$. If $\bar{q} \in \bar{U}$, we have $\bar{\sigma} \cap \partial \bar{U} \neq \emptyset$ which is a contradiction. This completes the proof.

Theorem 2. Let $\pi: \mathcal{X} \rightarrow X$ and $\pi_\mathcal{B}: \mathcal{B} \rightarrow D$ be covering maps where $\mathcal{B}$ is an open subset of $\mathcal{X}$. If $D$ is connected near $p \in \partial D$, then $p \in \pi(\partial \mathcal{B})$. 

PROOF. There is a neighborhood $W$ of $p$ which is evenly covered by $\pi$; that is, 
$\pi^{-1}(W) = \bigcup W_j$, where $\{W_j | j \in J\}$ is a disjoint family of open connected subsets 
and $\pi_j = \pi | W_j$; $W_j \rightarrow W$ is a homeomorphism for any $j \in J$. We can choose an open 
neighborhood $U$ of $p$ such that $U \subset W$ and $D \cap U$ is connected. Let $\tilde{U}_j = \pi_j^{-1}(U)$, 
$j \in J$. From $p \in \pi(\partial D)$, it is easily seen that $\tilde{D} \cap \pi^{-1}(U) \neq \emptyset$, so that $\tilde{D} \cap \tilde{U}_j \neq \emptyset$ 
for some $j$. Let $\tilde{p}_0 \in \tilde{D} \cap \tilde{U}_j$, then $\pi(\tilde{p}_0) \in D \cap U$. Let $\tilde{p} = \pi_j^{-1}(p)$. For an ar-
bitrary neighborhood $\tilde{V}$ of $\tilde{p}$ such that $\tilde{V} \subset \tilde{U}_j$, we put $V = \pi_j(\tilde{V})$. For a point 
$q \in D \cap V$, there exists an arc $\sigma$ in $D \cap U$ which joins $\pi(\tilde{p}_0)$ and $q$. Let $\tilde{\sigma} = \pi_j^{-1}(\sigma)$ 
and let $\tilde{q}$ be the terminal point of $\tilde{\sigma}$, then $\tilde{q} \in \tilde{V}$. On the other hand, $\tilde{\sigma}$ is lifted by 
$\pi_D$ to an arc $\tilde{\tau}$ in $\tilde{D}$ with the initial point $\tilde{p}_0$. From $\pi_\tilde{\sigma} = \pi_\tilde{\tau}$ follows that $\tilde{\sigma} = \tilde{\tau}$; this 
implies that $\tilde{q} \in \tilde{D} \cap \tilde{V}$. It is concluded that $\tilde{p} \in \partial \tilde{D}$ and $p = \pi(\tilde{p}) \in \pi(\partial D)$, which 
completes the proof.

Now, let $\pi: \tilde{X} \rightarrow X$ and $\pi_D: \tilde{D} \rightarrow D$ be covering maps and $D = \bigcup D_a$, $\tilde{D} = \bigcup \tilde{D}_i$ 
be decompositions; let $D$ be connected near the boundary. Let $I_a$ be as in the 
proof of Theorem 1, (2). Then, $\pi|\tilde{D}_i: \tilde{D}_i \rightarrow D_a$ is a covering map for every $i \in I_a$ 
and $D_a$ is connected near the boundary. $\tilde{D}$ is also connected near the boundary 
by Theorem 1, (1) and Lemma 3. Thus, we have the following.

**Corollary 3.** Let $\pi$, $\pi_D$, $\tilde{D}$, $D$ be as above. Then, $\pi(\partial \tilde{D}_i) = \partial D_a$, $i \in I_a$, 
and $\partial D = \bigcup \partial D_a$.

**Reference**