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Correspondence of Boundaries of Open Sets in Coverings

Nozomu Mochizuki*

1. Let \( \pi : \tilde{X} \to X \) be a covering map and \( \tilde{B} \) be an open subset of \( \tilde{X} \). The boundary of \( \tilde{B} \) will be denoted by \( \partial \tilde{B} \). Our main purpose is to discuss a relation between the condition \( \pi(\partial \tilde{B}) \subseteq \partial \pi(\tilde{B}) \) and a property of \( \pi|\tilde{B} \), the restriction of \( \pi \) to \( \tilde{B} \). Arguments used are quite elementary.

2. In what follows, all spaces are assumed to be locally arcwise connected Hausdorff spaces. If \( \tilde{X} \) is such a space, every open connected subset is arcwise connected; every component of an open subset is open. A continuous map \( \tilde{\sigma} : [0, 1] \to \tilde{X} \) with \( \tilde{\sigma}(0) = \tilde{p}, \tilde{\sigma}(1) = \tilde{q} \) will simply be called an arc \( \tilde{\sigma} \) in \( \tilde{X} \) with the initial point \( \tilde{p} \) and the terminal point \( \tilde{q} \); for this, we shall also say that \( \tilde{\sigma} \) joins \( \tilde{p} \) and \( \tilde{q} \).

**Lemma 1.** Let \( \pi : \tilde{X} \to X \) be a continuous map of \( \tilde{X} \) into \( X \) and let \( \tilde{D}, D \) be open subsets of \( \tilde{X}, X \), respectively, such that \( \pi(\tilde{D}) \subseteq D \). Then, \( \pi(\partial \tilde{D}) \subseteq \partial D \) if and only if every component of \( \tilde{D} \) is also a component of \( \pi^{-1}(\tilde{D}) \).

**Proof.** Let \( \pi(\partial \tilde{D}) \subseteq \partial D \). Let \( \tilde{D} = \bigcup_{i \in I} \tilde{D}_i \) be the decomposition of \( \tilde{D} \) into components where \( I \) denotes an index set. Since \( \partial \tilde{D} \subseteq \partial \tilde{D} \), we have \( \pi(\partial \tilde{D}_i) \subseteq \partial D \) for every \( i \in I \). The open subset \( \pi^{-1}(\tilde{D}) \) is decomposed into components such as \( \pi^{-1}(\tilde{D}) = \bigcup_{j \in J} \tilde{G}_j \) in which every \( \tilde{D}_i \) is contained in some \( \tilde{G}_j \). We suppose that \( \tilde{D}_i \nsubseteq \tilde{G}_j \) for some \( i \in I, j \in J \). Let \( \tilde{p} \in \tilde{D}_i, \tilde{q} \in \tilde{G}_j - \tilde{D}_i \). There exists an arc \( \tilde{\sigma} \) in \( \tilde{G}_j \) which joins \( \tilde{p} \) and \( \tilde{q} \). Since \( \tilde{\sigma} \) is connected, we have \( \tilde{\sigma} \cap \partial \tilde{D}_i \neq \emptyset \). It follows that \( D \cap \pi(\partial \tilde{D}_i) \neq \emptyset \); this is a contradiction.

3. In this section, \( \pi : \tilde{X} \to X \) denotes a continuous map of \( \tilde{X} \) onto \( X \). For an open subset \( \tilde{B} \) of \( \tilde{X} \) we shall put \( \pi_\tilde{B} = \pi|\tilde{B} \), the restriction of \( \pi \).

**Theorem 1.** (1) Let \( \pi : \tilde{X} \to X \) be a local homeomorphism, \( \tilde{B} \) be an open subset of \( \tilde{X} \). If \( \pi_\tilde{B} : \tilde{B} \to D \) is a covering map, then \( \pi(\partial \tilde{B}) \subseteq \partial D \).
(2) Let \( \pi: \bar{X} \to X \) be a covering map and let \( \bar{D}, D \) be open subsets of \( \bar{X}, X \) respectively such that \( \pi(\bar{D}) \subseteq \bar{D} \). If \( \pi(\partial \bar{D}) \subseteq \partial D \), then \( \pi_0: \bar{D} \to \pi(\bar{D}) \) is a covering map and \( \pi(\partial \bar{D}) \) lies dense in \( \partial \pi(\bar{D}) \).

**Proof.** (1) Suppose that \( \pi_0: \bar{D} \to D \) is a covering map. We must show that each component of \( \bar{D} \) is a component of \( \pi^{-1}(D) \). Let \( \bar{D}_0 \) be a component of \( \bar{D} \), then \( \bar{D}_0 \subseteq \bar{G}_0 \) for a component \( \bar{G}_0 \) of \( \pi^{-1}(D) \). We fix a point \( \bar{p} \in \bar{G}_0 \). For an arbitrary point \( \bar{q} \in \bar{G}_0 \), an arc \( \bar{\sigma} \) in \( \bar{G}_0 \) joins \( \bar{p} \) and \( \bar{q} \), and \( \pi \bar{\sigma} \) is an arc in \( D \). The covering map \( \pi_0: \bar{D} \to D \) lifts \( \pi \bar{\sigma} \) to an arc \( \bar{\tau} \) in \( \bar{D} \) with the initial point \( \bar{p} \). Since \( \bar{D}_0 \) is a component of \( \bar{D} \), \( \bar{\tau} \) is in \( \bar{D}_0 \). We have \( \pi \bar{\sigma} = \pi \bar{\tau} \) and since, as is well known, the lifting of an arc by a local homeomorphism is unique, it follows that \( \bar{\sigma} = \bar{\tau} \), so \( \bar{q} \in \bar{D}_0 \). Thus we have \( \bar{D}_0 = \bar{G}_0 \).

(2) Let \( \bar{D} = \bigcup \bar{D}_i \) and \( D = \bigcup D_a \) be decompositions into components. Let \( B = \{ \alpha \in A | \bar{D}_i \cap \pi^{-1}(D_a) \neq \emptyset \} \). Then, each \( \bar{D}_i \) is contained in some \( \pi^{-1}(D_a), \alpha \in B \). Let \( I_\alpha = \{ i \in l | \bar{D}_i \cap \pi^{-1}(D_a), \alpha \in B \} \), and let \( \bar{D}_\alpha = \bigcup_{i \in I_\alpha} \bar{D}_i \). Every \( \bar{D}_\alpha, i \in I_\alpha \), is a component of \( \pi^{-1}(D) \) from the assumption, hence a fortiori a component of \( \pi^{-1}(D_a) \). From this fact follows as usual that \( \pi(\bar{D}_\alpha) = D_a, i \in I_\alpha \), and \( \pi|\bar{D}_\alpha: \bar{D}_\alpha \to D_a \) is a covering map. It is clear that \( \bar{D} = \bigcup \bar{D}_\alpha, \pi(\bar{D}) = \bigcup D_a \), therefore, \( \pi_0: \bar{D} \to \pi(D) \) is a covering map. Now, let \( D_0 = \pi(D) \). Then, \( \pi(\partial \bar{D}) \subseteq \partial D_0 \) from the above. To see that \( \pi(\partial \bar{D}) \) is dense in \( \partial D_0 \), we suppose on the contrary that there exists \( p \in \partial D_0 \) such that \( p \notin \pi(\partial \bar{D}) \). Let \( U \) be an open connected neighborhood of \( p \) such that \( \pi(\partial \bar{D}) \cap U = \emptyset \). An arc \( \sigma \) in \( U \) joins \( p \) and a point \( q \in D_0 \cap U \). There exist \( \bar{q} \in \bar{D}, \pi(\bar{q}) = q \), and the lifting \( \bar{\sigma} \) of \( \sigma \) by \( \pi \) with the initial point \( \bar{q} \) and the terminal point \( \bar{p} \). From \( p \in \partial D_0 \) follows that \( \bar{p} \in \bar{D} \); hence, \( \bar{\sigma} \cap \partial \bar{D} \neq \emptyset \). This implies that \( \pi(\partial \bar{D}) \cap U \neq \emptyset \), a contradiction. The proof is completed.

**Corollary 1.** Let \( \pi: \bar{X} \to X \) be a covering map, \( \bar{D} \subseteq \bar{X} \) be an open subset and \( D = \pi(\bar{D}) \). Then \( \pi_0: \bar{D} \to D \) is a covering map if and only if \( \pi(\partial \bar{D}) \subseteq \partial D \).

The following Lemma 2 is essentially due to Corollary 1 to Theorem 7 in [1]; arguments there, combined with our Corollary 1, yield Corollary 2 below.

**Lemma 2.** Let \( \pi: \bar{X} \to X \) be a local homeomorphism and \( D = \pi(\bar{D}) \) where \( \bar{D} \) is an open subset of \( \bar{X} \). If \( \pi_0: \bar{D} \to D \) is a closed map, then \( \pi(\partial \bar{D}) \subseteq \partial D \).

**Proof.** Let \( \bar{p} \in \partial \bar{D}, p = \pi(\bar{p}) \). Choose open neighborhoods \( \bar{U}, U \) of \( \bar{p}, p \) respectively for which \( \pi|\bar{U}: \bar{U} \to U \) is a homeomorphism. There exists a net \( \{ \bar{p}_n \} \subseteq \bar{D} \cap \bar{U} \) such that \( \bar{p}_n \to \bar{p} \). The subset \( \pi(\bar{D} \cap \{ \bar{p}_n \}) \) is closed in \( D \) by assumption, and \( \pi(\bar{p}_n) \to p \). Let \( p \in D \), then \( p \) belongs to this subset; hence \( p = \pi(\bar{q}) \) for some \( \bar{q} \in \bar{D} \cap \{ \bar{p}_n \} \). If \( \bar{q} \in \bar{U} \), we have \( \bar{p} = \bar{q} \), a contradiction. On the other hand, if \( \bar{q} \in \bar{U} \), we have \( p \in \partial U \), since \( \bar{q} \in \partial \bar{U} \) and \( \pi(\partial \bar{U}) \subseteq \partial U \); this is also a contradiction. The proof is completed.
Corollary 2. Let $\tilde{B}$ be a relatively compact open subset of $\tilde{X}$ in Corollary 1, then the following are equivalent each other.

1. $\pi_\delta: \tilde{B} \to D$ is a covering map.
2. $\pi_\delta: \tilde{B} \to D$ is a closed map.
3. $\pi(\partial \tilde{B}) \subseteq \partial D$.
4. $\pi(\partial \tilde{B}) = \partial D$.

The relation $\pi(\partial \tilde{B}) = \partial D$ does not hold in general in Corollary 1. Let $\tilde{X}_n = \{(x, n) \in R^2|0 < x < 3\}$, $\tilde{X} = \bigcup_{n=1}^\infty \tilde{X}_n$, and $X = \{(x, 0) \in R^2|0 < x < 3\}$. By $\pi(x, y) = (x, 0)$, $\pi: \tilde{X} \to X$ is a covering map. Let $a_n = 2(1 - \frac{1}{2^n})$ and define $\tilde{B}_n = \{(x, n) \in R^2|a_{n-1} < x < a_n\}$, $\tilde{B} = \bigcup_{n=1}^\infty \tilde{B}_n$. Then, $D = \pi(\tilde{B}) = \{(x, 0)|0 < x < 2, x \neq a_n, n = 1, 2, \ldots\}$ and $\pi(x, 0) = \bigcup_{n=1}^\infty \{(a_n, 0)\}$ and $\partial D = \pi(\partial \tilde{B}) \cup \{(2, 0)\}$.

4. In this section, we shall introduce a concept of connectedness of an open set near a boundary point. Let $D$ be an open subset of $X$ and let $p \in \partial D$. We shall say that $D$ is connected near $p$ if the following is satisfied: For any neighborhood $U$ of $p$, there exists an open neighborhood $V$ of $p$ such that $V \subseteq U$ and $D \cap V$ is connected. If $D$ is connected near every boundary point, $D$ will be said to be connected near the boundary.

Let $D = \bigcup_{a \in A} D_a$ be the decomposition into components; let $p \in \partial D$. Then, $D$ is connected near $p$ if and only if there exists $a \in A$ such that $p \in \partial D_a$ where $D_a$ is connected near $p$ and $p \in \bigcup_{a \neq a} D_a$. In particular, if $D$ is connected near the boundary, we have $\partial D = \bigcup \partial D_a$, and every $D_a$ is connected near the boundary.

Lemma 3. Let $\pi: \tilde{X} \to X$ be a local homeomorphism and $\pi_\delta: \tilde{B} \to D$ be a covering map for an open subset $\tilde{B} \subseteq \tilde{X}$. Let $p \in \partial D$ and $p = \pi(p)$. If $D$ is connected near $p$, then $\tilde{B}$ is connected near $\tilde{p}$.

Proof. Let $\tilde{W}$ be a neighborhood of $\tilde{p}$. We can choose open neighborhoods $\tilde{U} \subseteq \tilde{W}$, $U$ of $\tilde{p}$, $p$ respectively such that $\pi(\tilde{U}) \to U$ is a homeomorphism and $D \cap U$ is connected. Clearly, we have $\pi(\partial \tilde{U}) \subseteq \partial U$. We shall see that $\pi|\tilde{B} \cap \tilde{U}: \tilde{B} \cap \tilde{U} \to D \cap U$ is a homeomorphism; for this, it is enough to see that $\pi|\tilde{B} \cap \tilde{U}$ is a map onto $D \cap U$. Let $\tilde{q}_0 \in \tilde{B} \cap \tilde{U}$. Clearly, $\pi(\tilde{q}_0) \in D \cap U$. For an arbitrary point $q \in D \cap U$, an arc $\sigma$ in $D \cap U$ joins $\pi(\tilde{q}_0)$ and $q$; this is lifted by $\pi_\delta$ to an arc $\tilde{\sigma}$ in $\tilde{B}$ with the initial point $\tilde{q}_0$ and the terminal point $\tilde{q} \in \tilde{B}$. If $\tilde{q} \not\in \tilde{U}$, we have $\tilde{\sigma} \cap \partial \tilde{U} = \emptyset$ which is a contradiction. This completes the proof.

Theorem 2. Let $\pi: \tilde{X} \to X$ and $\pi_\delta: \tilde{B} \to D$ be covering maps where $\tilde{B}$ is an open subset of $\tilde{X}$. If $D$ is connected near $p \in \partial D$, then $p \in \pi(\partial \tilde{B})$. 
PROOF. There is a neighborhood $W$ of $p$ which is evenly covered by $\pi$; that is, 
$\pi^{-1}(W) = \bigcup_{j \in J} \tilde{W}_j$, where $\{\tilde{W}_j \mid j \in J\}$ is a disjoint family of open connected subsets and $\pi_j = \pi|_{\tilde{W}_j}$: $\tilde{W}_j \rightarrow W$ is a homeomorphism for any $j \in J$. We can choose an open neighborhood $U$ of $p$ such that $U \subseteq W$ and $D \cap U$ is connected. Let $\tilde{U}_j = \pi^{-1}_j(U)$, $j \in J$. From $p \in \pi(\partial \tilde{D})$, it is easily seen that $\tilde{D} \cap \pi^{-1}(U) \neq \emptyset$, so that $\tilde{D} \cap \tilde{U}_j \neq \emptyset$ for some $j$. Let $\tilde{p}_0 \in \tilde{D} \cap \tilde{U}_j$, then $\pi(\tilde{p}_0) \in D \cap U$. Let $\tilde{p} = \pi^{-1}_j(p)$. For an arbitrary neighborhood $V$ of $\tilde{p}$ such that $\tilde{V} \subseteq \tilde{U}_j$, we put $V = \pi_j(\tilde{V})$. For a point $q \in D \cap V$, there exists an arc $\sigma$ in $D \cap U$ which joins $\pi(\tilde{p}_0)$ and $q$. Let $\tilde{\sigma} = \pi^{-1}_j(\sigma)$ and let $\tilde{q}$ be the terminal point of $\tilde{\sigma}$, then $\tilde{q} \in \tilde{V}$. On the other hand, $\tilde{\sigma}$ is lifted by $\pi_\tilde{D}$ to an arc $\tilde{\tau}$ in $\tilde{D}$ with the initial point $\tilde{p}_0$. From $\pi_\tilde{\sigma} = \pi_\tilde{\tau}$ follows that $\tilde{\sigma} = \tilde{\tau}$; this implies that $\tilde{q} \in \tilde{D} \cap \tilde{V}$. It is concluded that $\tilde{p} \in \partial \tilde{D}$ and $p = \pi(\tilde{p}) \in \pi(\partial \tilde{D})$, which completes the proof.

Now, let $\pi: \tilde{X} \rightarrow X$ and $\pi_\tilde{D}: \tilde{D} \rightarrow D$ be covering maps and $D = \bigcup_{a \in A} D_a$, $\tilde{D} = \bigcup_{i \in I} \tilde{D}_i$ be decompositions; let $D$ be connected near the boundary. Let $I_a$ be as in the proof of Theorem 1, (2). Then, $\pi|_{\tilde{D}_i}: \tilde{D}_i \rightarrow D_a$ is a covering map for every $i \in I_a$ and $D_a$ is connected near the boundary. $\tilde{D}$ is also connected near the boundary by Theorem 1, (1) and Lemma 3. Thus, we have the following.

**Corollary 3.** Let $\pi$, $\pi_\tilde{D}$, $\tilde{D}$, $D$ be as above. Then, $\pi(\partial \tilde{D}_i) = \partial D_a$, $i \in I_a$, and $\partial D = \bigcup_{a \in A} \partial D_a$.

**Reference**