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NOTES ON PRIMARY OR COPRIMARY MODULES

Ryûki Matsuda

This paper consists of three notes on primary or coprimary modules. In §1, we extend the result of [2] on primary ideals to primary submodules. In §2, we extend the result of Kirby[1] on coprimary modules to the case when the ring does not have the identity. In §3, we show that the \( \mathfrak{p} \)-primary for a maximal ideal \( \mathfrak{p} \) in the sense of Uda [3] is equivalent to the \( \mathfrak{p} \)-coprimary in essence.

1. A PROPERTY OF PRIMARY SUBMODULES. All definitions of terms in the following are of Zariski-Samuel [4] (Modules over a unitary ring are assumed unitary as usual). Let \( R \) be a (not necessarily with 1) commutative ring, \( \mathfrak{p} \) a maximal ideal of \( R \), \( \mathfrak{q} \) a \( \mathfrak{p} \)-primary ideal. Then [2] proved that every ideal \( \mathfrak{a} \) such that \( \mathfrak{p} \supset \mathfrak{a} \supset \mathfrak{q} \) is also \( \mathfrak{p} \)-primary. In this section, we generalize the result to submodules.

**Lemma 1.1** ([2], lemma). Let \( R \) be a commutative ring, \( \mathfrak{p} \) a maximal ideal of \( R \), \( \mathfrak{q} \) a \( \mathfrak{p} \)-primary ideal. Then there exist elements \( e \in R \) such that \( e^2 = e(\mathfrak{q}) \) and \( e \neq 0(\mathfrak{p}) \).

**Theorem 1.2.** Let \( R \) be a commutative ring, \( \mathfrak{p} \) a maximal ideal, \( M \) an \( R \)-module, \( N \) a \( \mathfrak{p} \)-primary submodule. Then every proper submodule containing \( N \) is also \( \mathfrak{p} \)-primary.

**Proof.** Let \( N' \) be a proper submodule of \( M \) containing \( N \). Since \( (N:M) = \mathfrak{q} \) is a \( \mathfrak{p} \)-primary ideal, there exists an element \( e \in R \) such that \( e^2 = e(\mathfrak{q}) \) and \( e \neq 0(\mathfrak{p}) \) by lemma 1.1. Suppose \( ax \in N' \) for \( a \in R \) and \( x \in M \). We will derive \( x \in \mathfrak{a} \) from the assumption \( a \notin \mathfrak{p} \). Since \( \mathfrak{p} \)...
is prime and maximal, we have \( aR + \mathfrak{p} = R \). There exist \( b \in R \) and \( p \in \mathfrak{p} \) such that \( ab + p = e \). Since \( N \) is \( \mathfrak{p} \)-primary, we have \( p^kM \subseteq N \) for some natural number \( k \). Multiplying \( k \)-times the both sides of \( ab + p = e \), we find \( b' \in R \) and \( q \in \mathfrak{p} \) such that \( ab' + p^k = e + q \). By \( ex + qx = b'(ax) + p^kx \), we have \( ex \in N' \). Since \( (e^2 - e)x \in N \), we have \( ex - x \in N \subseteq N' \), and hence \( x \in N' \). We have seen that \( N' \) is a primary submodule. It is obvious that the radical \( \sqrt{N'} : M \) of \( N' \) contains \( \mathfrak{p} \). If \( \sqrt{N'} : M \) contains \( \mathfrak{p} \) properly, \( \sqrt{N'} : M \) is \( R \) by the maximality of \( \mathfrak{p} \). Let \( x \) be any element of \( M \). We have \( e^{k'}x \in N' \) for some \( k' > 0 \). We can derive \( x \in N' \) by same way as the above argument. And there arises the contradiction of \( N' = M \). Therefore \( N' \) is a \( \mathfrak{p} \)-primary submodule of \( M \).

**PROPOSITION 1.3.** Let \( R \) be a commutative ring, \( \mathfrak{p} \) a maximal ideal of \( R \), \( \mathfrak{q} \) a \( \mathfrak{p} \)-primary ideal. Then every proper ideal containing \( \mathfrak{q} \) is also a \( \mathfrak{p} \)-primary ideal.

**COROLLARY 1.4.** ([2], proposition). Let \( R \) be a commutative ring, \( \mathfrak{p} \) a maximal ideal of \( R \), \( \mathfrak{q} \) a \( \mathfrak{p} \)-primary ideal. Then every ideal \( \mathfrak{a} \) such that \( \mathfrak{p} \supseteq \mathfrak{a} \supseteq \mathfrak{q} \) is also a \( \mathfrak{p} \)-primary.

**REMARK 1.5.1.** Let \( M \) be a module over a commutative ring \( R \). ([4] chap. 4, appendix says that \( \sqrt{N_1 + N_2} \) is equal to \( \sqrt{N_1} + \sqrt{N_2} \) for submodules \( N_1 \) and \( N_2 \). The assertion is false. The following is a counter example. We set \( R \) a commutative field, \( M = R \oplus R \) the direct sum, \( N_1 = R \oplus 0 \) and \( N_2 = 0 \oplus R \).

**REMARK 1.5.2.** Let \( M \) be a module over a commutative ring \( R \), \( N_1, \ldots, N_n \) a finite number of primary submodules belonging to maximal ideal \( \mathfrak{p} \). If \( N_1 + \ldots + N_n \) is distinct from \( M \), it is a \( \mathfrak{p} \)-primary submodule by theorem 1.2. But, as the example of above remark shows, this is not always the case.

**REMARK 1.5.3.** Let \( M \) be a module over a commutative ring \( R \). ([4] chap. 4, appendix says that, if the radical \( \sqrt{N} \)
of a submodule \( N \) is maximal, \( N \) is primary. It is true, if \( R \) has the identity. But it is not always true. For example, let \( X \) be a non-zero commutative additive group. Setting \( xy = 0 \) for every \( x, y \in X \), we have a ring \( A \). Let \( R = A \oplus F \) be the direct sum of rings \( A \) and a commutative field \( F \), \( M = R \) and \( N = 0 \).

2. A NOTE ON KIRBY'S PAPER. Let \( R \) be a commutative ring (not necessarily with 1), \( M \) an \( R \)-module. If \( a \notin \sqrt{0} : M \) implies \( aM = M \) for \( a \in R \), we call after Kirby [1] \( M \) a coprimary \( R \)-module. Then \( \sqrt{0} : M = \mathfrak{p} \) is a prime ideal of \( R \), and we call \( M \) a \( \mathfrak{p} \)-coprimary module. If \( M \) is the sum \( N_1 + \ldots + N_n \) of a finite number of coprimary submodules \( N_i \), the expression \( M = N_1 + \ldots + N_n \) is called coprimary decomposition. We set \( \sqrt{0} : N_i = \mathfrak{p}_i \). Then, if \( N_1 + \ldots + N_{i-1} + N_{i+1} + \ldots + N_n \neq M \) for \( i = 1, \ldots, n \), and \( \mathfrak{p}_1, \ldots, \mathfrak{p}_n \) are distinct each other, the expression is called a normal coprimary decomposition. Kirby proved the following: "let \( R \) be a commutative ring with 1 and with the maximum condition, \( M \) an \( R \)-module with the minimum condition with respect to submodules and with a coprimary decomposition \( M = N_1 + \ldots + N_n \), where \( N_i \) is a \( \mathfrak{p}_i \)-coprimary submodule for \( i = 1, \ldots, n \). If the \( \mathfrak{p}_i \) are distinct maximal ideals, then \( M \) has the maximum condition, \( M \) is the direct sum of the \( N_1, \ldots, N_n \) and \( M = N_1 + \ldots + N_n \) is the unique normal coprimary decomposition of \( M \)." In this section, we prove the same assertion for any commutative rings without the identities.

LEMMA 2.1. Let \( R \) be a commutative ring, \( \mathfrak{p} \) a maximal ideal, \( N \) an \( \mathfrak{p} \)-coprimary \( R \)-module. Then every proper submodule is also \( \mathfrak{p} \)-coprimary.

This is the dual analogue of the proposition of theorem 1.2.

LEMMA 2.2. ([4], chap. 3, theorem 21). Let \( M \) be a module over a commutative ring \( R \). Then a necessary and sufficient condition that \( M \) has a composition series is that it satisfies both the chain conditions.
LEMMA 2.3. ([1], theorem 2). Let $M$ be a module over a commutative ring $R$, $M = N_1 + \ldots + N_n$ and $M = N'_1 + \ldots + N'_m$ be two normal coprimary decompositions of $M$. We set $\sqrt{0}: N_i = p_i$ for $i = 1, \ldots, n$ and $\sqrt{0}: N'_j = p'_j$ for $j = 1, \ldots, m$. Then $n$ is equal to $m$ and the set $\{ p_1, \ldots, p_n \}$ is identical with $\{ p'_1, \ldots, p'_m \}$. Moreover, if $p_1 = p'_1$ is minimal among $\{ p_1, \ldots, p_n \}$, we have $N_1 = N'_1$.

THEOREM 2.4. Let $R$ be a commutative ring with the maximum condition, $M$ an $R$-module with the minimum condition and with a coprimary decomposition $M = N_1 + \ldots + N_n$, where $N_i$ is a $p_i$-coprimary submodule for $i = 1, \ldots, n$. If the $p_i$ are distinct maximal ideals of $R$, then $M$ has the maximum condition, $M$ is the direct sum of the $N_i$ and $M = N_1 + \ldots + N_n$ is the unique normal coprimary decomposition of $M$.

PROOF. Suppose $N_1 \cap (N_1 + \ldots + N_{i-1} + N_{i+1} + \ldots + N_n) = N$ is not zero for some $i$, say $i = 1$. Since $\sqrt{0}: N_1 = p_1$, we see $\sqrt{0}: N_2 + \ldots + N_n \supseteq p_2 \ldots p_n$. Since $\sqrt{0}: N = p_1$ by lemma 2.1, we have $p_1 \supseteq \sqrt{0}: N_2 + \ldots + N_n \supseteq p_2 \ldots p_n$.

Since $p_1$ is prime, at least one of $\{ p_2, \ldots, p_n \}$ is contained in $p_1$. This contradicts with the assumption that the $p_i$ are maximal. Therefore we see $N = 0$, and hence $M$ is the direct sum of the $N_i$. Therefore $N_i$ is not contained in $N_1 + \ldots + N_{i-1} + N_{i+1} + \ldots + N_n$ for $i = 1, \ldots, n$, hence $M = N_1 + \ldots + N_n$ is a normal coprimary decomposition. Since each $p_i$ is minimal among $\{ p_1, \ldots, p_n \}$, $M = N_1 + \ldots + N_n$ is the unique normal decomposition by lemma 2.3.

Since $R$ has the maximum condition and $p_1$ is $\sqrt{0}: N_1$, we have $p_1^{k(i)} N_1 = 0$ for some natural number $k(i)$ for $i = 1, \ldots, n$, hence $p_1^{k(1)} \ldots p_1^{k(n)} M = 0$. As [4], chap. 4, 2 does, we consider the sequence

$$M \supset p_1 M \supset p_1^2 M \supset \ldots \supset p_1^{k(1)} M \supset p_1^{k(1)} p_2 M \supset \ldots \supset p_1^{k(1)} \ldots p_n^{k(n)} M = 0.$$ 

We see that $R/p_1$ are fields for $i = 1, \ldots, n$. Therefore
each consecutive two terms have composition series, hence $M$ has the maximum condition by lemma 2.2. We have proved all the assertions of theorem 2.4.

3. THE MAXIMAL IDEAL-PRIMARY IN THE SENSE OF UDA. Let $R$ be a (not necessarily with 1) commutative ring, $\mathfrak{p}$ an ideal of $R$, $M$ an $R$-module. When $(0 : x)$ is a $\mathfrak{p}$-primary ideal for every $0 \neq x \in M$, Uda [3] calls $M$ to be a $\mathfrak{p}$-primary module. In this section, we show that $M$ is $\mathfrak{p}$-primary for a maximal $\mathfrak{p}$ in the sense of Uda if and only if $Rx$ is $\mathfrak{p}$-coprimary for every $0 \neq x \in M$.

**Lemma 3.1.** Let $R$ be a commutative ring, $\mathfrak{p}$ a maximal ideal of $R$, $\mathfrak{q}$ a $\mathfrak{p}$-primary ideal. Then $\mathfrak{p}$ is the unique maximal ideal containing $\mathfrak{q}$.

This is contained in proposition 1.3.

**Proposition 3.2.** Let $R$ be a commutative ring, $\mathfrak{p}$ a maximal ideal, $M$ an $R$-module. If $M$ is $\mathfrak{p}$-primary in the sense of Uda, $Rx$ is $\mathfrak{p}$-coprimary for every $0 \neq x \in M$.

**Proof.** Let $x$ be any non-zero element of $M$. $(0 : x) = \mathfrak{q}$ is a $\mathfrak{p}$-primary ideal. We see therefore $\sqrt{0 : Rx} = \mathfrak{p}$. Taking an element $a \notin \mathfrak{p}$, we consider an ideal $\mathfrak{u} = aR + \mathfrak{q}$. Since $\mathfrak{q} \subseteq a \subseteq \mathfrak{p}$, we have $\mathfrak{u} = R$ by lemma 3.1. Therefore for every $b \in R$, there exist $c \in R$ and $q \in \mathfrak{q}$ such that $b = ac + q$. Since $bx = acx + qx = acx$, we see $Rx \subseteq aRx$. We have seen that $Rx$ is $\mathfrak{p}$-coprimary.

**Proposition 3.3.** Let $R$ be a commutative ring, $\mathfrak{p}$ an ideal of $R$, $M$ an $R$-module. If $Rx$ is $\mathfrak{p}$-coprimary for every $0 \neq x \in M$, then $M$ is $\mathfrak{p}$-primary in the sense of Uda.

**Proof.** Let $x$ be any non-zero element of $M$. We denote $(0 : x)$ by $\mathfrak{q}$. Since $\sqrt{0 : Rx} = \mathfrak{p}$, we see $\mathfrak{q} \subseteq \mathfrak{p}$. It is easy to see that $\mathfrak{p}$ is contained in $\sqrt{\mathfrak{q}}$. Suppose $ab \in \mathfrak{q}$ and $b \notin \mathfrak{q}$ for $a, b$ of $R$. Since $bx \neq 0$, we have $\sqrt{0 : Rbx} = \mathfrak{p}$. Since $abx = 0$, $a$ belongs to $\mathfrak{p}$. Therefore $\mathfrak{q}$ is $\mathfrak{p}$-primary.
By propositions 3.2 and 3.3, we have

**THEOREM 3.4.** Let $R$ be a commutative ring, $\mathfrak{q}$ a maximal ideal, $M$ an $R$-module. Then $M$ is $\mathfrak{q}$-primary in the sense of Uda if and only if $Rx$ is $\mathfrak{q}$-coprimary for every $0 \neq x \in M$.

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