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NOTES ON PRIMARY OR COPRIMARY MODULES

Ryûki Matsuda

This paper consists of three notes on primary or coprimary modules. In §1, we extend the result of [2] on primary ideals to primary submodules. In §2, we extend the result of Kirby[1] on coprimary modules to the case when the ring does not have the identity. In §3, we show that the \( p \)-primary for a maximal ideal \( p \) in the sense of Uda [3] is equivalent to the \( p \)-coprimary in essence.

1. A PROPERTY OF PRIMARY SUBMODULES. All definitions of terms in the following are of Zariski-Samuel [4] (Modules over a unitary ring are assumed unitary as usual). Let \( R \) be a (not necessarily with 1) commutative ring, \( p \) a maximal ideal of \( R \), \( q \) a \( p \)-primary ideal. Then [2] proved that every ideal \( a \) such that \( p \supseteq a \supseteq q \) is also \( p \)-primary. In this section, we generalize the result to submodules.

LEMMA 1.1 ([2], lemma). Let \( R \) be a commutative ring, \( p \) a maximal ideal of \( R \), \( q \) a \( p \)-primary ideal. Then there exist elements \( e \in R \) such that \( e^2 = e(q) \) and \( e \notin 0(p) \).

THEOREM 1.2. Let \( R \) be a commutative ring, \( p \) a maximal ideal, \( M \) an \( R \)-module, \( N \) a \( p \)-primary submodule. Then every proper submodule containing \( N \) is also \( p \)-primary.

PROOF. Let \( N' \) be a proper submodule of \( M \) containing \( N \). Since \( (N:M) = q \) is a \( p \)-primary ideal, there exists an element \( e \in R \) such that \( e^2 = e(q) \) and \( e \notin 0(p) \) by lemma 1.1. Suppose \( ax \in N' \) for \( a \in R \) and \( x \in M \). We will derive \( x \in a \) from the assumption \( a \notin p \). Since \( p \)
is prime and maximal, we have $aR + \mathfrak{p} = R$. There exist $b \in R$ and $p \in \mathfrak{p}$ such that $ab + p = e$. Since $N$ is $\mathfrak{p}$-primary, we have $p^kM \subset N$ for some natural number $k$. Multiplying $k$-times the both sides of $ab + p = e$, we find $b' \in R$ and $q \in \mathfrak{p}$ such that $ab' + p^k = e + q$. By $ex + qx = b'(ax) + p^kx$, we have $ex \in N'$. Since $(e^2 - e)x \in N$, we have $ex - x \in N \subset N'$, and hence $x \in N'$. We have seen that $N'$ is a primary submodule. It is obvious that the radical $\sqrt{N}: \mathfrak{m}$ of $N'$ contains $\mathfrak{p}$. If $\sqrt{N}: \mathfrak{m}$ contains $\mathfrak{p}$ properly, $\sqrt{N}: \mathfrak{m}$ is $R$ by the maximality of $\mathfrak{p}$. Let $x$ be any element of $M$. We have $e^{k'}x \in N'$ for some $k' > 0$. We can derive $x \in N'$ by the same way as the above argument. And there arises the contradiction of $N' = M$. Therefore $N'$ is a $\mathfrak{p}$-primary submodule of $M$.

**PROPOSITION 1.3.** Let $R$ be a commutative ring, $\mathfrak{p}$ a maximal ideal of $R$, $\mathfrak{q}$ a $\mathfrak{p}$-primary ideal. Then every proper ideal containing $\mathfrak{q}$ is also a $\mathfrak{p}$-primary ideal.

**COROLLARY 1.4.** ([2], proposition). Let $R$ be a commutative ring, $\mathfrak{p}$ a maximal ideal of $R$, $\mathfrak{q}$ a $\mathfrak{p}$-primary ideal. Then every ideal $\mathfrak{a}$ such that $\mathfrak{p} \supset \mathfrak{a} \supset \mathfrak{q}$ is also $\mathfrak{p}$-primary.

**REMARK 1.5.1.** Let $M$ be a module over a commutative ring $R$. [4] chap. 4, appendix says that $\sqrt{N_1 + N_2}$ is equal to $\sqrt{N_1} + \sqrt{N_2}$ for submodules $N_1$ and $N_2$. The assertion is false. The following is a counter example. We set $R$ a commutative field, $M = R \oplus R$ the direct sum, $N_1 = R \oplus 0$ and $N_2 = 0 \oplus R$.

**REMARK 1.5.2.** Let $M$ be a module over a commutative ring $R$, $N_1, \ldots, N_n$ a finite number of primary submodules belonging to maximal ideal $\mathfrak{p}$. If $N_1 + \ldots + N_n$ is distinct from $M$, it is $\mathfrak{p}$-primary submodule by theorem 1.2. But, as the example of above remark shows, this is not always the case.

**REMARK 1.5.3.** Let $M$ be a module over a commutative ring $R$. [4] chap. 4, appendix says that, if the radical $\sqrt{\mathfrak{p}}$
of a submodule $N$ is maximal, $N$ is primary. It is true, if $R$ has the identity. But it is not always true. For example, let $X$ be a non-zero commutative additive group. Setting $xy = 0$ for every $x, y \in X$, we have a ring $A$. Let $R = A \oplus F$ be the direct sum of rings $A$ and a commutative field $F$, $M = R$ and $N = 0$.

2. A NOTE ON KIRBY'S PAPER. Let $R$ be a commutative ring (not necessarily with 1), $M$ an $R$-module. If $a \in R$ implies $aM = M$ for $a \in R$, we call after Kirby [1] $M$ a coprimary $R$-module. Then $\sqrt{0 : M} = \mathfrak{p}$ is a prime ideal of $R$, and we call $M$ a $\mathfrak{p}$-coprimary module. If $M$ is the sum $N_1 + \ldots + N_n$ of a finite number of coprimary submodules $N_i$, the expression $M = N_1 + \ldots + N_n$ is called coprimary decomposition. We set $\sqrt{0 : N_i} = \mathfrak{p}_i$. Then, if $N_1 + \ldots + N_{i-1} + N_{i+1} + \ldots + N_n \neq M$ for $i = 1, \ldots, n$, and $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are distinct each other, the expression is called a normal coprimary decomposition. Kirby proved the following: "let $R$ be a commutative ring with 1 and with the maximum condition, $M$ an $R$-module with the minimum condition with respect to submodules and with a coprimary decomposition $M = N_1 + \ldots + N_n$, where $N_i$ is a $\mathfrak{p}_i$-coprimary submodule for $i = 1, \ldots, n$. If the $\mathfrak{p}_i$ are distinct maximal ideals, then $M$ has the maximum condition, $M$ is the direct sum of the $N_1, \ldots, N_n$ and $M = N_1 + \ldots + N_n$ is the unique normal coprimary decomposition of $M". In this section, we prove the same assertion for any commutative rings without the identities.

**Lemma 2.1.** Let $R$ be a commutative ring, $\mathfrak{p}$ a maximal ideal, $N$ a $\mathfrak{p}$-coprimary $R$-module. Then every proper submodule is also $\mathfrak{p}$-coprimary.

This is the dual analogue of the proposition of theorem 1.2.

**Lemma 2.2.** ([4], chap. 3, theorem 21). Let $M$ be a module over a commutative ring $R$. Then a necessary and sufficient condition that $M$ has a composition series is that it satisfies both the chain conditions.
LEMMA 2.3. ([1], theorem 2). Let $M$ be a module over a commutative ring $R$, $M = N_1 + \ldots + N_n$ and $M = N'_1 + \ldots + N'_m$ two normal coprimary decompositions of $M$. We set $\sqrt{0}: N_1 = \mathfrak{p}_1$ for $i = 1, \ldots, n$ and $\sqrt{0}: N'_j = \mathfrak{p}'_j$ for $j = 1, \ldots, m$. Then $n$ is equal to $m$ and the set $\{ \mathfrak{p}_1, \ldots, \mathfrak{p}_n \}$ is identical with $\{ \mathfrak{p}'_1, \ldots, \mathfrak{p}'_m \}$. Moreover, if $\mathfrak{p}_i = \mathfrak{p}'_j$ is minimal among $\{ \mathfrak{p}_1, \ldots, \mathfrak{p}_n \}$, we have $N_i = N'_j$.

THEOREM 2.4. Let $R$ be a commutative ring with the maximum condition, $M$ an $R$-module with the minimum condition and with a coprimary decomposition $M = N_1 + \ldots + N_n$, where $N_i$ is a $\mathfrak{p}_i$-coprimary submodule for $i = 1, \ldots, n$. If the $\mathfrak{p}_i$ are distinct maximal ideals of $R$, then $M$ has the maximum condition, $M$ is the direct sum of the $N_i$ and $M = N_1 + \ldots + N_n$ is the unique normal coprimary decomposition of $M$.

PROOF. Suppose $N_1 \cap (N_1 + \ldots + N_{1-1} + N_{1+1} + \ldots + N_n) = N$ is not zero for some $i$, say $i = 1$. Since $\sqrt{0}: N_1 = \mathfrak{p}_1$, we see $\sqrt{0}: N_2 + \ldots + N_n \supset \mathfrak{p}_2 \ldots \mathfrak{p}_n$. Since $\sqrt{0}: N = \mathfrak{p}_1$ by lemma 2.1, we have $\mathfrak{p}_1 \supset \sqrt{0}: N_2 + \ldots + N_n \supset \mathfrak{p}_2 \ldots \mathfrak{p}_n$.

Since $\mathfrak{p}_1$ is prime, at least one of $\{ \mathfrak{p}_2, \ldots, \mathfrak{p}_n \}$ is contained in $\mathfrak{p}_1$. This contradicts with the assumption that the $\mathfrak{p}_i$ are maximal. Therefore we see $N = 0$, and hence $M$ is the direct sum of the $N_i$. Therefore $N_i$ is not contained in $N_1 + \ldots + N_{i-1} + N_{i+1} + \ldots + N_n$ for $i = 1, \ldots, n$, hence $M = N_1 + \ldots + N_n$ is a normal coprimary decomposition.

Since each $\mathfrak{p}_i$ is minimal among $\{ \mathfrak{p}_1, \ldots, \mathfrak{p}_n \}$, $M = N_1 + \ldots + N_n$ is the unique normal decomposition by lemma 2.3.

Since $R$ has the maximum condition and $\mathfrak{p}_1$ is $\sqrt{0}: N_1$, we have $\mathfrak{p}_1^{k(1)} N_1 = 0$ for some natural number $k(1)$ for $i = 1, \ldots, n$, hence $\mathfrak{p}_1^{k(1)} \ldots \mathfrak{p}_n^{k(n)} M = 0$. As [4], chap. 4, 2 does, we consider the sequence

$$M \supset \mathfrak{p}_1 M \supset \mathfrak{p}_1^2 M \supset \ldots \supset \mathfrak{p}_1^{k(1)} M \supset \mathfrak{p}_1^{k(1)} \mathfrak{p}_2 M$$

$$\supset \ldots \supset \mathfrak{p}_1^{k(1)} \ldots \mathfrak{p}_n^{k(n)} M = 0.$$

We see that $R/\mathfrak{p}_i$ are fields for $i = 1, \ldots, n$. Therefore
each consecutive two terms have composition series, hence $M$ has the maximum condition by lemma 2.2. We have proved all the assertions of theorem 2.4.

3. THE MAXIMAL IDEAL-PRIMARY IN THE SENSE OF UDA. Let $R$ be a (not necessarily with 1) commutative ring, $\mathfrak{p}$ an ideal of $R$, $M$ an $R$-module. When $(0 : x)$ is a $\mathfrak{p}$-primary ideal for every $0 \neq x \in M$, Uda [3] calls $M$ to be a $\mathfrak{p}$-primary module. In this section, we show that $M$ is $\mathfrak{p}$-primary for a maximal $\mathfrak{m}$ in the sense of Uda if and only if $Rx$ is $\mathfrak{m}$-coprimary for every $0 \neq x \in M$.

**LEMMA 3.1.** Let $R$ be a commutative ring, $\mathfrak{p}$ a maximal ideal of $R$, $\mathfrak{q}$ a $\mathfrak{p}$-primary ideal. Then $\mathfrak{p}$ is the unique maximal ideal containing $\mathfrak{q}$.

This is contained in proposition 1.3.

**PROPOSITION 3.2.** Let $R$ be a commutative ring, $\mathfrak{p}$ a maximal ideal, $M$ an $R$-module. If $M$ is $\mathfrak{p}$-primary in the sense of Uda, $Rx$ is $\mathfrak{p}$-coprimary for every $0 \neq x \in M$.

**PROOF.** Let $x$ be any non-zero element of $M$. $(0 : x) = \mathfrak{q}$ is a $\mathfrak{p}$-primary ideal. We see therefore $\sqrt{0 :Rx} = \mathfrak{p}$. Taking an element $a \notin \mathfrak{p}$, we consider an ideal $\mathfrak{u} = aR + \mathfrak{q}$. Since $\mathfrak{q} \subset a \subset \mathfrak{p}$, we have $\mathfrak{u} = R$ by lemma 3.1. Therefore for every $b \in R$, there exist $c \in R$ and $q \in \mathfrak{q}$ such that $b = ac + q$. Since $bx = acx + qx = acx$, we see $Rx \subset aRx$. We have seen that $Rx$ is $\mathfrak{p}$-coprimary.

**PROPOSITION 3.3.** Let $R$ be a commutative ring, $\mathfrak{p}$ an ideal of $R$, $M$ an $R$-module. If $Rx$ is $\mathfrak{p}$-coprimary for every $0 \neq x \in M$, then $M$ is $\mathfrak{p}$-primary in the sense of Uda.

**PROOF.** Let $x$ be any non-zero element of $M$. We denote $(0 : x)$ by $\mathfrak{q}$. Since $\sqrt{0 : Rx} = \mathfrak{p}$, we see $\mathfrak{q} \subset \mathfrak{p}$. It is easy to see that $\mathfrak{p}$ is contained in $\sqrt{\mathfrak{q}}$. Suppose $ab \in \mathfrak{q}$ and $b \notin \mathfrak{p}$ for $a, b$ of $R$. Since $bx \neq 0$, we have $\sqrt{0 : Rbx} = \mathfrak{p}$. Since $abx = 0$, $a$ belongs to $\mathfrak{p}$. Therefore $\mathfrak{q}$ is $\mathfrak{p}$-primary.
By propositions 3.2 and 3.3, we have

**THEOREM 3.4.** Let $R$ be a commutative ring, $\mathfrak{m}$ a maximal ideal, $M$ an $R$-module. Then $M$ is $\mathfrak{m}$-primary in the sense of Uda if and only if $Rx$ is $\mathfrak{m}$-coprimary for every $0 \neq x \in M$.

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