A NOTE ON THE IDENTITIES OF DEDEKIND RINGS

Author(s)
MATSUDA, Ryuki

Citation
Bulletin of the Faculty of Science, Ibaraki University. Series A, Mathematics, 6: 1-2

Issue Date
1974

URL
http://hdl.handle.net/10109/2857

Rights
このリポジトリに収録されているコンテンツの著作権は、それぞれの著作権者に帰属します。引用、転載、複製等される場合は、著作権法を遵守してください。
A NOTE ON THE IDENTITIES OF DEDEKIND RINGS

Ryûki Matsuda

It has been proved that, if a commutative ring $R$ without zero-divisors satisfies the condition of Dedekind rings, $R$ is the usual Dedekind ring with the identity. In this note, we extend the assertion and prove the following:

**THEOREM.** Let $R$ be a commutative ring all elements of which are not zero-divisors. If every ideal of $R$ is a product of prime ideals ($\neq R$), $R$ has the identity.

**PROOF.** Let $K$ be the total quotient ring of $R$. We denote $(R:R)_K$ by $R^{-1}$. If $aR^{-1}$ coincides with $R$ for all non-zero-divisors $a$ of $R$, we have

$$R = a^2R^{-1} = aR.$$

Then we have $a = ab$ for some $b$ of $R$. We see that $b$ is the identity of $R$. We may suppose therefore that $dR^{-1}$ is a proper subset of $R$ for some non-zero-divisor $d$ of $R$. Since $dR^{-1}$ is an ideal of $R$, there exist prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ such that $dR^{-1} = \mathfrak{p}_1 \ldots \mathfrak{p}_n$ ($n \geq 1$). There exist prime ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$ also such that $\mathfrak{p}_nR = \mathfrak{q}_1 \ldots \mathfrak{q}_m$. Every $\mathfrak{p}_i$ contains $\mathfrak{q}_n$ and $\mathfrak{p}_n$ contains $\mathfrak{q}_j$ for some $j$, say $j = 1$. If $\mathfrak{p}_nR$ is a proper subset of $\mathfrak{p}_n$, we have

$$\mathfrak{p}_nR = \mathfrak{p}_n \mathfrak{q}_2 \ldots \mathfrak{q}_m$$

($m \geq 2$).

Multiplying $d^{-1}R \cdot \mathfrak{p}_1 \ldots \mathfrak{p}_{n-1}$ on the both sides, we have

$$R^2 = R \mathfrak{q}_2 \ldots \mathfrak{q}_m.$$

There arises the contradiction of $R \subset \mathfrak{q}_m$. Therefore $\mathfrak{p}_nR$ coincides with $\mathfrak{p}_n$. Therefore we have

$$dR^{-1}R = \mathfrak{p}_1 \ldots \mathfrak{p}_nR = \mathfrak{p}_1 \ldots \mathfrak{p}_n = dR^{-1}.$$
Therefore we see that $R^{-1}R$ coincides with $R^{-1}$. Since $R \supseteq R^{-1}R$ and $1 \in R^{-1}$ by the definition, we see that $R$ contains the identity.

REMARK. There exist rings $R$ which satisfy the conditions of the theorem and has zero-divisors. $\mathbb{Z}/(4)$ is one of the examples.

Ryûki Matsuda
Department of Mathematics,
Faculty of Science,
Ibaraki University, Mito.

(Received October 25, 1973)