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A lower bound of nilpotency class of the group of self-homotopy classes

Hideaki Ōshima

We work in the category of path-connected topological spaces with nondegenerate base points. If $G$ is a topological group (or a group-like space) whose unit is the base point and $X$ is a space, then the set $[X, G]$ of homotopy classes of based maps from $X$ to $G$ inherits a group structure from $G$. We have been interested in the group $H(G) = [G, G]$. Let $\text{nil}(\Gamma)$ be the nilpotency class of a group $\Gamma$ and $X \times n$ the product space $X \times \cdots \times X$ ($n$ factors) for an integer $n \geq 1$. The purpose of this note is to prove

Proposition. If $G$ is not homotopy nilpotent, then

$$\text{nil}(H(G \times n)) \geq \max\{n, \text{nil}(H(G))\}. \quad (1)$$

It follows from [5, 6, 8] that a connected Lie group is not homotopy nilpotent if and only if it has torsion in homology. Hence, for example, we have $\text{nil}(H(G_2 \times n)) \geq n$ from (1), where $G_2$ is the exceptional Lie group of rank 2. Notice that $G_2 \times n$ is simply connected and not simple when $n \geq 2$. On the other hand, the main theorem of [1] says that $\text{nil}(H(\text{PU}(p))) \geq n$ provided $p$ is a prime number $\geq n + 2$. Here PU($p$) is the projective unitary group of order $p$ and so it is simple and not simply connected.

Let $T^n(X)$ be the subspace of $X^n$ consisting of all points, at least one of whose coordinates is the base point. Let $j_n : T^n(X) \to X^n$ be the inclusion map, $d_{n,X} : X \to X^n$ the diagonal map, and $\text{cat}(X)$ the Lusternik-Schnirelmann category of $X$ defined by Whitehead [2]. Then $\text{cat}(X) < n$ if and only if there is a map $\varphi : X \to T^n(X)$ such that $j_n \circ \varphi$ is homotopic to $d_{n,X}$. Let $c_n : G^n \to G$ be the iterated commutator map, that is, $c_1 = \text{id}_G$ (the identity map), $c_2(x, y) = [x, y] = xxy^{-1}y^{-1}$, and $c_n = c_2 \circ (c_{n-1} \times \text{id}_G)$ for $n \geq 3$. We define $\text{hpnil}(G)$ to be the least integer $n$ such that $c_{n+1}$ is nullhomotopic. If there is no such integer, we define $\text{hpnil}(G) = \infty$. Then $G$ is homotopy nilpotent if and only if $\text{hpnil}(G)$ is finite.

Given maps $f_1, f_2, \ldots, f_n : X \to G$, we have

$$[[\ldots [[f_1, f_2], f_3], \ldots], f_n] = c_n \circ (f_1 \times \cdots \times f_n) \circ d_{n,X}. \quad (2)$$

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Hence we readily obtain the following well-known inequality [7].
\[
\text{nil}([X,G]) \leq \min \{ \text{cat}(X), \text{hpnil}(G) \}.
\] (3)

Since \([X,G^x] \cong [X,G] \oplus \cdots \oplus [X,G] (n \text{ factors})\), we have
\[
\text{nil}([X,G^x]) = \text{nil}([X,G]).
\] (4)

Since \(\text{cat}(X \times Y) \leq \text{cat}(X) + \text{cat}(Y)\) [2], it follows that \(\text{cat}(G^x) \leq n \cdot \text{cat}(G)\). Let \(\text{pr}_i : G^x \to G\) be the projection to the \(i\)-th factor (\(1 \leq i \leq n\)). Since \(\text{pr}_i^* : \mathcal{H}(G) \to \mathcal{H}(G^x)\) is a monomorphism, we have \(\text{nil}(\mathcal{H}(G)) \leq \text{nil}(\mathcal{H}(G^x))\). Hence we have
\[
\text{nil}(\mathcal{H}(G)) \leq \text{nil}(\mathcal{H}(G^x)) \leq \min \{ n \cdot \text{cat}(G), \text{hpnil}(G) \}.
\] (5)

**Proof of Proposition.** Suppose that \(G\) is not homotopy nilpotent. It suffices to prove \(\text{nil}(\mathcal{H}(G^x,G)) \geq n\) because of (4) and (5). By the assumption, \(c_n\) is essential for every \(n \geq 1\). When \(n = 1\), it means that \(\mathcal{H}(G)\) is not trivial, that is, \(\text{nil}(\mathcal{H}(G)) \geq 1\). If \(n \geq 2\), then \([\ldots [[\text{pr}_1, \text{pr}_2], \text{pr}_3], \ldots, \text{pr}_{n-1}], \text{pr}_n] = c_n\) by (2) and so \(\text{nil}(\mathcal{H}(G^x,G)) \geq n\). \(\square\)

**Remark.** Since \(\text{cat}(S^{3 \times 3}) = \text{hpnil}(S^3) = \text{nil}(\mathcal{H}(S^{3 \times 3})) = 3\) for all \(n \geq 3\) by [4, §4], it follows that the assumption “not homotopy nilpotent” is essential in Proposition and that (3) is best possible.

**References**


