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An estimate for derivative of the de la Vallée Poussin mean

Kentaro Itoh*, Ryozi Sakai** and Noriaki Suzuki***

Abstract

The de la Vallée Poussin mean for exponential weights on $(-\infty, \infty)$ was investigated in [6]. In the present paper we discuss its derivatives. An estimate for the Christoffel function plays an important role.

1. Introduction

Let $\mathbb{R} = (-\infty, \infty)$. We consider an exponential weight

$$w(x) = \exp(-Q(x))$$

on $\mathbb{R}$, where $Q$ is an even and nonnegative function on $\mathbb{R}$. Throughout this paper we always assume that $w$ belongs to a relevant class $\mathcal{F}(C^2+)$ (see section 2). A function $T = T_w$ defined by

$$T(x) := \frac{xQ'(x)}{Q(x)}, \quad x \neq 0$$

is very important. We call $w$ a Freud-type weight if $T$ is bounded, and otherwise, $w$ is called an Erdős-type weight. For $x > 0$, the Mhaskar-Rakhmanov-Saff number (MRS number) $a_x = a_x(w)$ of $w = \exp(-Q)$ is defined by a positive root of the equation

$$x = \frac{2}{\pi} \int_0^1 \frac{a_x uQ'(a_x u)}{(1 - u^2)^{1/2}} du.$$  

When $w = \exp(-Q) \in \mathcal{F}(C^2+)$, $Q'$ is positive and increasing on $(0, \infty)$, so that

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(1.3) \[ \lim_{x \to \infty} a_x = \infty \quad \text{and} \quad \lim_{x \to -\infty} a_x = 0 \]
and
(1.4) \[ \lim_{x \to \infty} \frac{a_x}{x} = 0 \quad \text{and} \quad \lim_{x \to +0} \frac{a_x}{x} = \infty \]
hold. Note that those convergences are all monotonically.

Let \( \{p_n\} \) be orthogonal polynomials for a weight \( w \), that is, \( p_n \) is the polynomial of degree \( n \) such that
\[
\int_{\mathbb{R}} p_n(x)p_m(x)w(x)^2dx = \delta_{mn}.
\]

Note that when \( w(x) = \exp(-|x|^2) \), then \( \{p_n\} \) are Hermite polynomials.

For \( 1 \leq p \leq \infty \), we denote by \( L^p(I) \) the usual \( L^p \) space on an interval \( I \) in \( \mathbb{R} \). For a function \( f \) with \( fw \in L^p(\mathbb{R}) \), we set
\[
s_n(f)(x) := \sum_{k=0}^{n-1} b_k(f)p_k(x) \quad \text{where} \quad b_k(f) = \int_{\mathbb{R}} f(t)p_k(t)w(t)^2dt
\]
for \( n \in \mathbb{N} \) (the partial sum of Fourier series). The de la Vallée Poussin mean \( v_n(f) \) of \( f \) is defined by
\[
v_n(f)(x) := \frac{1}{n} \sum_{j=n+1}^{2n} s_j(f)(x).
\]

In [6], we proved the following: Let \( 1 \leq p \leq \infty \) and \( w \in \mathcal{F}(C^2+) \). Assume that
(1.5) \[ T(a_n) \leq C \left( \frac{n}{a_n} \right)^{2/3} \]
for some \( C > 1 \). Then there exists another constant \( C > 1 \) such that if \( fw \in L^p(\mathbb{R}) \), then
(1.6) \[ \|v_n(f)w\|_{L^p(\mathbb{R})} \leq C\|fw\|_{L^p(\mathbb{R})} \]
holds for all \( n \in \mathbb{N} \), and if \( T^{1/4}fw \in L^p(\mathbb{R}) \), then
(1.7) \[ \|v_n(f)w\|_{L^p(\mathbb{R})} \leq C\|T^{1/4}fw\|_{L^p(\mathbb{R})} \]
holds for all \( n \in \mathbb{N} \). It is also known that
(1.8) \[ \|P'w\|_{L^p(\mathbb{R})} \leq C \left( \frac{n}{a_n} \right) \|Pw\|_{L^p(\mathbb{R})}, \]
for all \( P \in \mathcal{P}_n \), where \( \mathcal{P}_n \) is the set of all polynomials of degree at most \( n \) (see [5, Theorem 6.1]). Since \( v_n(f) \in \mathcal{P}_{2n-1} \), combining (1.7) with (1.8), we have
(1.9) \[ \|v_n'(f)w\|_{L^p(\mathbb{R})} \leq C \left( \frac{n}{a_n} \right) \|T^{1/4}fw\|_{L^p(\mathbb{R})} \]
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with some $C > 1$. Here we use the fact that $a_n$ and $a_{2n}$ are comparable (see Lemma 2.1 (1) below). The inequality (1.9) suggests us the following: if $fw \in L^p(\mathbb{R})$, then

$$(1.10) \quad \left\| v'_n(f) \frac{w}{T^{3/4}} \right\|_{L^p(\mathbb{R})} \leq C \left( \frac{n}{a_n} \right) \| fw \|_{L^p(\mathbb{R})}$$

and, if $T^{3/4}fw \in L^p(\mathbb{R})$, then

$$(1.11) \quad \| v'_n(f)w \|_{L^p(\mathbb{R})} \leq C \left( \frac{n}{a_n} \right) \| T^{3/4}fw \|_{L^p(\mathbb{R})}$$

holds.

In the present paper, we will show that (1.10) holds for all $1 \leq p \leq \infty$ and (1.11) is true for $2 \leq p \leq \infty$ at the least. More generally, as for the $j$th derivative $v^{(j)}_n(f)$ of $v_n(f)$, the following theorems are established.

**Theorem 1.1.** Let $k \geq 2$ be an integer and let $w \in \mathcal{F}_\lambda(C^4 +)$ with $0 < \lambda < (k + 3)/(k + 2)$, and let $1 \leq p \leq \infty$. Then there exists a constant $C > 1$ such that if $1 \leq j \leq k$, and if $fw \in L^p(\mathbb{R})$, then

$$(1.12) \quad \left\| v^{(j)}_n(f) \frac{w}{T^{(2j+1)/4}} \right\|_{L^p(\mathbb{R})} \leq C \left( \frac{n}{a_n} \right)^j \| fw \|_{L^p(\mathbb{R})}$$

holds for all $n \in \mathbb{N}$.

The definition of a class $\mathcal{F}_\lambda(C^4 +)$ is given in section 2.

**Theorem 1.2.** Let $k$ and $w$ be as in Theorem 1.1, and let $2 \leq p \leq \infty$. Then there exists a constant $C > 1$ such that if $1 \leq j \leq k$, and if $T^{(2j+1)/4}fw \in L^p(\mathbb{R})$, then

$$(1.13) \quad \| v^{(j)}_n(f)w \|_{L^p(\mathbb{R})} \leq C \left( \frac{n}{a_n} \right)^j \| T^{(2j+1)/4}fw \|_{L^p(\mathbb{R})}$$

holds for all $n \in \mathbb{N}$.

**Theorem 1.3.** Let $k$ and $w$ be as in Theorem 1.1, and let $1 \leq p \leq 2$. Then there exists a constant $C > 1$ such that for every $1 \leq j \leq k$ and every $T^{(2j+1)/4}fw \in L^2(\mathbb{R})$, we have

$$(1.14) \quad \| v^{(j)}_n(f)w \|_{L^p(\mathbb{R})} \leq C \left( \frac{n}{a_n} \right)^j a_n^{(2-p)/2p} \| T^{(2j+1)/4}fw \|_{L^2(\mathbb{R})}$$

for all $n \in \mathbb{N}$.

We note that when $w$ is a Freud-type weight, then $1 \leq T(x) \leq C$, so that,

$$(1.15) \quad \| v^{(j)}_n(f)w \|_{L^p(\mathbb{R})} \leq C \left( \frac{n}{a_n} \right)^j \| fw \|_{L^p(\mathbb{R})}$$
follows from Theorem 1.1. In [3, Chapter 3], Mhaskar discussed the first derivative of the de la Vallée Poussin mean for Freud-type weights. Our contribution is to deal with not only Freud-type but also Erdős-type weights. In the proofs of above theorems, we use Mhaskar’s argument. In addition, there are two keys: one is to use mollification of exponential weights (see Lemma 2.4 below) which was obtained in [5], and another is to estimate the Christoffel functions which are done in section 3. Unfortunately, we do not know whether (1.13) holds true or not for $1 \leq p < 2$, however, we will give another estimate which holds for all $1 \leq p \leq \infty$ in section 4. A related inequality to (1.14) is also given in section 6.

Throughout this paper, we write $f(x) \sim g(x)$ for a subset $I \subset \mathbb{R}$ if there exists a constant $C \geq 1$ such that $f(x)/C \leq g(x) \leq Cf(x)$ holds for all $x \in I$. Similarly, $a_n \sim b_n$ means that $a_n/C \leq b_n \leq Ca_n$ holds for all $n \in \mathbb{N}$. We will use the same letter $C$ to denote various positive constants; it may vary even within a line. Roughly speaking, $C > 1$ implies that $C$ is sufficiently large, and differently, $C > 0$ means $C$ is a sufficiently small positive number.

2. Definitions and Lemmas

We say that an exponential weight $w = \exp(-Q)$ belongs to class $\mathcal{F}(C^2+)$, when $Q : \mathbb{R} \to [0, \infty)$ is a continuous and even function and satisfies the following conditions:

(a) $Q'(x)$ is continuous in $\mathbb{R}$ and $Q(0) = 0$.
(b) $Q''(x)$ exists and is positive in $\mathbb{R} \setminus \{0\}$.
(c) $\lim_{x \to \infty} Q(x) = \infty$.
(d) The function $T$ in (1.1) is quasi-increasing in $(0, \infty)$ (i.e. there exists $C > 1$ such that $T(x) \leq CT(y)$ whenever $0 < x < y$), and there exists $\Lambda \in \mathbb{R}$ such that $T(x) \geq \Lambda > 1$, $x \in \mathbb{R} \setminus \{0\}$.

(e) There exists $C > 1$ such that

$$\frac{Q''(x)}{|Q'(x)|} \leq C \frac{|Q'(x)|}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R}.$$ 

There also exist a compact subinterval $J(\ni 0)$ of $\mathbb{R}$, and $C > 1$ such that

$$C \frac{Q''(x)}{|Q'(x)|} \geq \frac{|Q'(x)|}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R} \setminus J.$$ 

Let $\lambda > 0$. We write $w \in \mathcal{F}_\lambda(C^2+)$ if there exist $K > 1$ and $C > 1$ such that for all $|x| \geq K$,

$$\frac{|Q'(x)|}{Q(x)^{\lambda}} \leq C \tag{2.1}$$

holds. We also write $w \in \mathcal{F}_\lambda(C^3+)$, if $Q \in C^3(\mathbb{R} \setminus \{0\})$ and

$$\frac{|Q^{(3)}(x)|}{Q''(x)} \leq C \frac{|Q''(x)|}{Q'(x)} \quad \text{and} \quad \frac{|Q'(x)|}{Q(x)^{\lambda}} \leq C$$
hold for every $|x| \geq K$. Moreover, we write $w \in \mathcal{F}\lambda(C^4+)$, if $Q \in C^4(\mathbb{R} \setminus \{0\})$ and

$$
\frac{|Q^{(3)}(x)|}{Q''(x)} \sim \frac{|Q''(x)|}{Q'(x)}, \quad \left| \frac{Q^{(4)}(x)}{Q^{(3)}(x)} \right| \leq C \left| \frac{Q''(x)}{Q'(x)} \right| \quad \text{and} \quad \frac{|Q'(x)|}{Q(x)} \leq C
$$

hold for every $|x| \geq K$. Clearly $\mathcal{F}\lambda(C^4+)$ \ (see \[4, Proposition 3 (3.8)\]).

A typical example of Freud-type weight is $w(x) = \exp(-|x|^\alpha)$ with $\alpha > 1$. It belongs to $\mathcal{F}\lambda(C^4+)$ for $\lambda = 1$. For $u \geq 0$, $\alpha > 0$ with $\alpha + u > 1$ and $l \in \mathbb{N}$, we set

$$
Q(x) := |x|^u(\exp(|x|^\alpha) - \exp(0)),
$$

where $\exp_l(x) := \exp(\exp(\cdots (\exp(x)\cdots)))$ ($l$-times). Then $w(x) = \exp(-Q(x))$ is an Erdös-type weight, which belongs to $\mathcal{F}\lambda(C^4+)$ for any $\lambda > 1$ (see\[1\]).

In the following lemmas we fix $w \in \mathcal{F}(C^2+)$.

**Lemma 2.1.** Fix $L > 0$. Then we have

1. (1) $a_t \sim a_{Lt}$ on $t > 0$ (see \[2, Lemma 3.5 (a)\]).
   2. (2) $Q(a_t) \sim Q(a_{Lt}), Q'(a_t) \sim Q'(a_{Lt})$ and $T(a_{Lt}) \sim T(a_t)$ on $t > 0$ (see \[2, Lemma 3.5 (b)\]).
   3. (3) $\frac{1}{T(a_t)} \sim 1 - \frac{a_{Lt}}{a_t}$ on $t > 0$ (see \[2, Lemma 3.11 (3.52)\]).
   4. (4) $\frac{t}{\sqrt{T(a_t)}} \sim Q(a_t)$ and $\frac{t\sqrt{T(a_t)}}{a_t} \sim |Q'(a_t)|$ on $t > 0$ (see \[2, Lemma 3.4 (3.18)\] and $3.17\)\).
   5. (5) Assume that $w$ is an Erdös-type weight. Then for every $\eta > 0$, there exists a constant $C_\eta > 1$ such that

$$
a_x \leq C_\eta x^\eta \quad (x \geq 1)
$$

(see \[4, Proposition 3 (3.8)\]).

**Lemma 2.2.** (\[2, Theorem 1.9 (a)\]) Let $1 \leq p \leq \infty$. Then

$$
\|Pw\|_{L^p(\mathbb{R})} \leq 2\|Pw\|_{L^p([-a_n,a_n])}
$$

for every $n \in \mathbb{N}$ and every $P \in \mathcal{P}_n$.

**Lemma 2.3.** (1) There exist constants $C_1 > 1$ and $c_0 > 0$ such that if $|x - t| < c_0/T(x)$ then $T(t)/C_1 \leq T(x) \leq C_1T(t)$ holds (cf. \[2, Theorem 3.2 (e)\] see also \[6, Lemma 3.4\]).
   2. (2) There exist a constant $C_2 > 1$ such that for any $n \in \mathbb{N}$, if $|t|, |x| < a_{2n}$ and $|x - t| \leq a_n/n$ then $T(t)/C_2 \leq T(x) \leq C_2T(t)$ holds (see \[6, (4.6)\]).

**Lemma 2.4.** (\[5, Theorem 4.1 and (4.11)\]) Let $m = 1, 2$ and let $w \in \mathcal{F}\lambda(C^{2+m}+)$ with $0 < \lambda < (m + 2)/(m + 1)$. For every $\alpha \in \mathbb{R}$, we can construct a new weight $w^* \in \mathcal{F}\lambda(C^{1+m}+)$ such that

$$
w^*(x) \sim T(x)^{\alpha}w(x) \quad \text{and} \quad T^*(x) \sim T(x)
$$
on $\mathbb{R}$, and

\[(a_{x/c})^\ast \leq a_x^* \leq a_{cx}\]

holds on $\mathbb{R}$ with some constant $c > 1$, where $T^*$ and $a_x^*$ are corresponding ones defined in (1.1) and (1.2) with respect to a weight $w^*$ respectively.

Using the above lemma, we obtain the following assertions. First one is a generalization of (1.8). Second assertion was shown in [5, Corollary 6.2] under some additional assumption.

\textbf{Lemma 2.5.} Let $w \in \mathcal{F}_\lambda(C^3+)$ with $0 < \lambda < 3/2$ and let $1 \leq p \leq \infty$. For $j \in \mathbb{N}$, there exists a constant $C_1 > 1$ such that for every $n \in \mathbb{N}$ and every $P \in \mathcal{P}_n$, we have

\[(2.6) \quad \left\| \frac{P(j)}{T^{j/2}} \right\|_{L^p(\mathbb{R})} \leq C_3 \left( \frac{n}{a_n} \right)^j \| Pw \|_{L^p(\mathbb{R})}\]

and if we further assume that $w \in \mathcal{F}_\lambda(C^4+)$ with $0 < \lambda < 4/3$, then there exists a constant $C_4 > 1$ such that

\[(2.7) \quad \left\| P(j) w \right\|_{L^p(\mathbb{R})} \leq C_4 \left( \frac{n}{a_n} \right)^j \| T^{j/2} Pw \|_{L^p(\mathbb{R})}\]

also holds.

\textbf{Proof.} For $i = 1, \ldots, j$, let $w^*_j \in \mathcal{F}_\lambda(C^2+)$ be a weight obtained in Lemma 2.4 for $\alpha = -(i - 1)/2$. Then, since $P(j) \in \mathcal{P}_{n-j}$, by (1.8) for $w^*_j$ and by (2.4) and (2.5), there exists a constant $C > 1$ such that

\[\left\| \frac{P(j)}{T^{j/2}} \right\|_{L^p(\mathbb{R})} \leq C \left( \frac{n - j + 1}{a(a_n+1/c)} \right)^j \left\| P(j-1) w^*_j \right\|_{L^p(\mathbb{R})}.\]

Since $w^*_j(x) \sim T(x)^{-1/2} w^*_j-1(x)$, we also see

\[\left\| \frac{P(j)}{T^{j/2}} \right\|_{L^p(\mathbb{R})} \leq C \left( \frac{n - j + 1}{a(a_n+1/c)} \right)^j \left\| P(j-1) w^*_j-1 \right\|_{L^p(\mathbb{R})}.\]

Repeating this process, we have

\[\left\| \frac{P(j)}{T^{j/2}} \right\|_{L^p(\mathbb{R})} \leq C \left\| \frac{P(j)}{T^{j/2}} \right\|_{L^p(\mathbb{R})} \leq C^j \left( \frac{n + j + 1}{a(a_n+1/c)} \right)^j \right\| Pw \|_{L^p(\mathbb{R})}\]

where we use Lemma 2.1 (1).
For (2.7), we first remark that if \( w \in \mathcal{F}_x(C^3) \), then

\[
\|P'w\|_{L^p(\mathbb{R})} \leq C_4 \left( \frac{n}{a_n} \right) \|T^{1/2}Pw\|_{L^p(\mathbb{R})}
\]

holds true (see [5, (1.4) and its proof]). This is the case \( j = 1 \). To show general case \( j > 1 \), we consider a weight \( w_i^* \in \mathcal{F}_x(C^3) \) in Lemma 2.4 for \( w \in \mathcal{F}_x(C^4) \) with \( \alpha = (i - 1)/2 \) \( (i = 1, \cdots, j) \). Applying \( P^{(j-1)} \) and \( w_i^* \) to (2.8) and repeating this process for \( i = 1, \cdots, j \), we obtain (2.7) as in (2.6). This completes the proof.

**Lemma 2.6.** Let \( k \in \mathbb{N} \cup \{0\} \) and \( w \in \mathcal{F}_x(C^2) \) with \( 0 < \lambda < (k + 2)/(k + 1) \). Then there exist constants \( C_5 > 1 \) and \( \delta > 0 \) such that

\[
T(a_n) \leq C_5 n^{2/(2k+3) - \delta}
\]

holds for all \( n \in \mathbb{N} \).

Proof. We may assume that \( w = \exp(-Q) \) is an Erd"os-type weight. By (2.1), \( |Q'(x)|/Q(x)^\lambda \leq C \) with some constant \( C > 1 \). Hence Lemma 2.1 (4) gives us

\[
\frac{n \sqrt{T(a_n)}}{a_n} \left( \frac{n}{\sqrt{T(a_n)}} \right)^{-\lambda} \leq C,
\]

that is, \( T(a_n) \leq C n^{2/(\lambda+1)} a_n^{2(\lambda-1)/(\lambda+1)} \). Since \( \lambda < (k + 2)/(k + 1) \), we can choose \( \delta > 0 \) and \( \eta > 0 \) such that \( 2(\lambda - 1)/(\lambda + 1) + \delta + 2\eta < 2/(2k + 3) \). Hence (2.9) follows from Lemma 2.1 (5). This completes the proof.

We remark that (2.9) implies (1.5). Hence if \( w \in \mathcal{F}_x(C^2) \) with \( 0 < \lambda < (k + 2)/(k + 1) \), then (1.6), (1.7), (1.8) and (1.9) hold true.

**Lemma 2.7.** Let \( w \in \mathcal{F}_x(C^2) \) with \( 0 < \lambda < 2 \). Then there exists a constant \( C_6 > 1 \) such that for every \( n \in \mathbb{N} \), if \( |t|, |x| < a_{2n} \) and if \( |t - x| < a_n/(n \sqrt{T(x)}) \) then

\[
w(t)/C_6 \leq w(x) \leq C_6 w(t)
\]

Proof. By Lemma 2.3 (2), we have \( T(t)/C_2 \leq T(x) \leq C_2 T(t) \), and by (1.3) we can write \( |t| = a_n \). Then \( a_n \leq a_{2n} \) implies \( s \leq 2n \). Hence (1.4) and Lemma 2.1(1) show \( s a_n/(a_s) \leq C_7 \) with some constant \( C_7 > 1 \). Since \( |Q'(t)| \leq C s \sqrt{T(a_s)/a_s} \) by Lemma 2.1 (4), we have

\[
|Q'(t)||t - x| \leq C s \sqrt{T(a_s)/a_s} \frac{a_n}{a_s} \frac{1}{n \sqrt{T(x)}} \leq C C_7 \frac{a_n}{n} \frac{s \sqrt{T(t)}}{a_s \sqrt{T(x)}} \leq C C_7 \sqrt{C_2}.
\]

Similarly, we see \( |Q'(x)||t - x| \leq C C_7 \). Hence if we put \( C_6 = e^{C C_7 \sqrt{C_2}} \), then \( |Q'(t)||t - x| \leq \log C_6 \) and \( |Q'(x)||t - x| < \log C_6 \) hold true. From the mean value theorem for
differential calculus, there exists $\theta$ between $x$ and $t$ such that

$$\frac{w(x)}{w(t)} = \exp(Q(t) - Q(x)) = \exp(Q'(\theta)(t - x)).$$

Since $Q'$ is increasing, $|Q'(\theta)(x - t)| \leq \max\{|Q'(x)|, |Q'(t)|\}|x - t| \leq \log C_0$, which shows (2.10) immediately. This completes the proof.

3. Estimates for Christoffel functions

By definition, the partial sum of Fourier series is given by

$$(3.1) \quad s_n(f)(x) = \int_{\mathbb{R}} K_n(x, t) f(t) w(t)^2 dt,$$

where

$$K_n(x, t) = \sum_{k=0}^{n-1} p_k(x) p_k(t).$$

It is known that by the Cristoffel-Darboux formula

$$(3.3) \quad K_n(x, t) = \frac{\gamma_n - 1}{\gamma_n} \frac{p_n(x)p_{n-1}(t) - p_n(t)p_{n-1}(x)}{x - t}$$

holds, where $\gamma_n$ and $\gamma_{n-1}$ are the leading coefficients of $p_n$ and $p_{n-1}$, respectively. Then

$$(3.4) \quad a_n \sim \frac{\gamma_{n-1}}{\gamma_n}$$

also holds (see [2, Lemma 13.9]).

The Christoffel function $\lambda_n(x) = \lambda_n(w, x)$ is defined by

$$\lambda_n(x) := \frac{1}{K_n(x, x)} = \left(\sum_{k=0}^{n-1} p_k(x)^2\right)^{-1}.$$

Then

$$(3.5) \quad \lambda_n(x) = \inf_{P \in P_{n-1}} \frac{1}{P(x)^2} \int_{\mathbb{R}} |P(t)w(t)|^2 dt,$$

holds on $\mathbb{R}$. We use derivative versions of (3.5). The following equality is also established.

**Proposition 3.1.** Let $0 \leq j < n$. Then for every $x \in \mathbb{R}$, we have

$$(3.6) \quad \left(\sum_{k=0}^{n-1} (p_k^{(j)}(x))^2\right)^{-1} = \inf_{P \in P_{n-1}} \frac{1}{(P^{(j)}(x))^2} \int_{\mathbb{R}} |P(t)w(t)|^2 dt.$$
Proof. In [3, Theorem 1.3.2], we see
\[
\left( \sum_{k=0}^{n-1} \phi(p_k)^2 \right)^{-1} = \inf_{P \in \mathcal{P}_{n-1}} \frac{1}{\Phi(P)^2} \int_{\mathbb{R}} |P(t)w(t)|^2 dt
\]
for any linear functional \( \Phi \) on polynomials. (3.6) follows if we consider \( \Phi(P) = P^{(j)}(x) \).

The following estimate plays an important role in our later argument. We use \( C_m \) (\( m = 1, \cdots, 6 \)), which are constants in lemmas of the previous section.

**Proposition 3.2.** Let \( k \geq 2 \) be an integer and let \( w \in \mathcal{F}_\lambda(C^4_{\ast}) \) with \( 0 < \lambda < (k + 3)/(k + 2) \). Then there exists a constant \( C_8 > 1 \) such that for every \( 1 \leq j \leq k \) and every \( n \in \mathbb{N} \),
\[
(3.7) \quad \frac{w(x)^2}{T(x)^{(2j+1)/2}} \sum_{k=0}^{n-1} (p_k^{(j)}(x))^2 \leq C_8 \left( \frac{n}{a_n} \right)^{2j+1}.
\]

Proof. It is enough to show (3.7) for sufficiently large \( n \). By Proposition 3.1, (3.7) follows from
\[
(3.8) \quad \left( \frac{a_n}{n} \right)^{2j+1} \frac{w(x)^2}{T(x)^{(2j+1)/2}} \leq C_8 \frac{1}{(P^{(j)}(x))^2} \int_{\mathbb{R}} |P(t)w(t)|^2 dt
\]
for \( P \in \mathcal{P}_{n-1} \). Now to show (3.8), take \( P \in \mathcal{P}_{n-1} \) arbitrarily. By Lemma 2.2, we can choose \( a \in \mathbb{R} \) such that \( |a| \leq a_{n-1} \) and satisfies
\[
(3.9) \quad \|wP\|_{L^\infty(\mathbb{R})} \leq 2|w(\zeta)P(\zeta)|.
\]

Let \( 0 < c_1 \leq 1 \). Lemma 2.6 gives us \( T(a_n) \leq C_5 n^{1-\delta'} \) with some \( \delta' > 0 \), so that if \( t \in \mathbb{R} \) satisfies
\[
(3.10) \quad |t - \zeta| \leq c_1 \frac{a_n}{n} \frac{1}{\sqrt{T(\zeta)}},
\]
then
\[
|t| \leq |\zeta| + |\zeta - t| \leq |\zeta| + c_1 \frac{a_n}{n} \frac{1}{\sqrt{T(\zeta)}} \leq a_{n-1} + \frac{a_n}{n} \leq a_n + \frac{C_5}{n^{\delta'}} \frac{a_n}{T(a_n)}.
\]
Since there exists a constant \( C > 1 \) such that \( a_n + a_n/(CT(a_n)) \leq a_{2n} \) by Lemma 2.1 (3), if we take \( n_0 \in \mathbb{N} \) such that \( n_0^{\delta'} > CC_5 \), then
\[
(3.11) \quad |t| \leq a_{2n}
\]
for all \( n \geq n_0 \). Hence by Lemma 2.7, \( w(t)/C_6 \leq w(\zeta) \leq C_6 w(t) \) holds. By monotonicity of \( w \), \( w(u)/C_6 \leq w(\zeta) \leq C_6 w(u) \) also holds for every \( u \) between \( t \) and \( \zeta \). Moreover,
since $T$ is quasi-increasing, Lemma 2.3 (2) shows $\sqrt{T(u)} \leq C \sqrt{T(\zeta)}$ with some $C > 1$. Then using (2.6) for $p = \infty$ and $j = 1$, we have

$$|P(\zeta)| - |P(t)| \leq |P(t) - P(\zeta)| = \left| \int_{\zeta}^{t} P'(u)du \right|$$

$$\leq CC_6 \frac{\sqrt{T(\zeta)}}{w(\zeta)} \left| \int_{\zeta}^{t} \frac{1}{\sqrt{T(u)}} w(u)P'(u)du \right|$$

$$\leq CC_6 |t - \zeta| \frac{\sqrt{T(\zeta)}}{w(\zeta)} \left\| \frac{w}{\sqrt{T}} P' \right\|_{L^\infty(\mathbb{R})}$$

$$\leq CC_6 C_3 |t - \zeta| \frac{\sqrt{T(\zeta)}}{w(\zeta)} \frac{n}{a_n} \|wP\|_{L^\infty(\mathbb{R})}$$

$$\leq 2c_1 CC_6 C_3 |P(\zeta)|$$

by (3.9) and (3.10). Consequently, if we take $c_1 > 0$ so small that $2c_1 CC_6 C_3 < 1/2$, we have

$$|P(t)| \geq \frac{1}{2} |P(\zeta)| \text{ if } |t - \zeta| \leq c_1 \frac{a_n}{n} \frac{1}{\sqrt{T(\zeta)}}. \tag{3.12}$$

Since $C_2 T(t) \geq T(\zeta)$ and $C_6 w(t) \geq w(\zeta)$, (3.9) and (3.12) show

$$\int_{\mathbb{R}} \sqrt{T(\zeta)}|P(t)|^2 w(t)^2 dt \geq \frac{\sqrt{T(\zeta)}}{\sqrt{C_2}} \int_{|t - \zeta| \leq c_1 a_n/(n \sqrt{T(\zeta)})} |P(t)|^2 w(t)^2 dt$$

$$\geq \frac{\sqrt{T(\zeta)}}{\sqrt{C_2}} |P(\zeta)|^2 \frac{w(\zeta)^2}{4} \frac{a_n}{C_6} \frac{1}{n} \frac{\sqrt{T(\zeta)}}{a_n}$$

$$\geq \frac{c_1}{4 \sqrt{C_2}} \frac{a_n}{C_6} \frac{\|wP\|_{L^\infty(\mathbb{R})}}{n}$$

$$=: \frac{1}{C_0} \frac{a_n}{n} \|wP\|_{L^\infty(\mathbb{R})}^2.$$
and hence by (2.4) and (2.5) we see

\[
\int_{\mathbb{R}} |P(t)|^2 w^2(t) \, dt \geq \frac{1}{C} \int_{\mathbb{R}} \sqrt{T^*(t)} |P(t)|^2 w^*(t)^2 \, dt
\]

\[
\geq \frac{1}{CC_0C_3} \left( \frac{a_n}{n} \right)^{2j+1} \frac{w^*(x)^2}{T^*(x)^j} |P(j)(x)|^2
\]

\[
\geq \frac{1}{C} \left( \frac{a_n}{n} \right)^{2j+1} \frac{w(x)^2}{T(x)^{(2j+1)/2}} |P(j)(x)|^2.
\]

This together with Lemma 2.1 (1) shows (3.8) and the proof is completed.

4. Proof of Theorem 1.1

In the remaining sections, we again use \(C_m (m = 1, \ldots, 6)\) without notice, which are constants in lemmas in section 2.

Let \(1 \leq p \leq \infty, k \geq 2, w \in F_3(C^{4+})\) with \(0 < \lambda < (k+3)/(k+2)\) and let \(1 \leq j \leq k\). Due to Lemma 2.4, there is \(w^* \in F_3(C^{4+})\) such that \(w^*(x) \sim T(x)^{-2j+1/4} w(x)\).

Take \(fw \in L^p(\mathbb{R})\) arbitrarily. Since \(v_n^{(j)}(f) \in P_{2n-1-j}\), applying \(w^*\) to (2.7), we have

\[
\left\| v_n^{(j)}(f) w^* \right\|_{L^p(\mathbb{R})} \leq C \left\| v_n^{(j)}(f) w^* \right\|_{L^p(\mathbb{R})}
\]

\[
\leq C \left( \frac{2n - j}{a_{2n-j}} \right)^j \left\| (T^*)^{j/2} v_n(f) w^* \right\|_{L^p(\mathbb{R})}
\]

\[
\leq C \left( \frac{a_{2n-j}}{a_n} \right)^j \left\| v_n(f) w \right\|_{L^p(\mathbb{R})}
\]

\[
\leq C \left( \frac{a_k}{a_n} \right)^j \left\| fw \right\|_{L^p(\mathbb{R})}.
\]

Here we use Lemma 2.1 (1), (2.4) and (2.5). The last inequality follows from (1.6).

This completes the proof of Theorem 1.1.

By a similar argument as above, we also have

\[
\left\| v_n^{(j)}(f) w \right\|_{L^p(\mathbb{R})} \leq C \left( \frac{a_k}{a_n} \right)^j T(a_n)^{(2j+1)/4} \left\| fw \right\|_{L^p(\mathbb{R})}
\]

for all \(1 \leq p \leq \infty\). In fact, take \(w^* \in F_3(C^{3+})\) such that \(w^*(x) \sim T^{j/2}(x) w(x)\). Then
by (2.7) for \( w \) and by Lemma 2.4 and Lemma 2.2 for \( w^* \), we have

\[
\|v_n^{(j)}(f)w\|_{L^p(\mathbb{R})} \leq C \left( \frac{n}{a_n} \right)^j \|T^{j/2}v_n(f)w\|_{L^p(\mathbb{R})}
\]

\[
\leq C \left( \frac{n}{a_n} \right)^j \|v_n(f)w^*\|_{L^p([-a_{2n}, a_{2n}])}
\]

\[
\leq C \left( \frac{n}{a_n} \right)^j \left\| v_n(f)T^{(2j)+1/4} \frac{w}{T^{1/4}} \right\|_{L^p([-a_{2n}, a_{2n}])}
\]

\[
\leq C \left( \frac{n}{a_n} \right)^j T(a_n)^{(2j)+1/4} \left\| v_n(f) \frac{w}{T^{1/4}} \right\|_{L^p([-a_{2n}, a_{2n}])}
\]

\[
\leq C \left( \frac{n}{a_n} \right)^j T(a_n)^{(2j)+1/4} \|fw\|_{L^p(\mathbb{R})}.
\]

Note that by Lemma 2.1 (2), \( T(x) \leq CT(a_{2cn}) \leq CT(a_n) \) holds for all \( x \in [-a_{2cn}, a_{2cn}] \), because \( T \) is quasi-increasing.

5. Proof of Theorem 1.2

Let \( k \geq 2 \), \( w \in \mathcal{F}_k(C^4+) \) with \( 0 < \lambda < (k + 3)/(k + 2) \) and let \( 1 \leq j \leq k \). We first show (1.13) for the case \( p = \infty \). Suppose that \( T^{(2j)+1/4}fw \in L^\infty(\mathbb{R}) \). Since \( v_n^{(j)}(f) \in \mathcal{P}_{2n} \), by Lemma 2.2, it is sufficient to show

\[
|v_n^{(j)}(f)(x)w(x)| \leq C \left( \frac{n}{a_n} \right)^j \|T^{(2j)+1/4}fw\|_{L^\infty(\mathbb{R})}
\]

for every \( |x| \leq a_{2n} \). Now we set

\[
A_n := \{ t \in \mathbb{R}; |t - x| < \frac{a_{2n}}{2n} \}, \quad B_n := \{ t \in \mathbb{R}; \frac{a_{2n}}{2n} \leq |t - x| < \frac{c_0}{T(x)} \}
\]

and \( C_n := \mathbb{R} \setminus (A_n \cup B_n) \), where \( c_0 > 0 \) is a constant in Lemma 2.3 (1). Then as in the proof of (3.11), there exists \( n_0 \in \mathbb{N} \) such that if \( n \geq n_0 \) and \( t \in A_n \), then \( |t| \leq a_{4n} \) holds. Hence Lemma 2.3 (2) implies

\[
T(t)/C_2 \leq T(x) \leq C_2T(t) \quad (t \in A_n).
\]

Since \( T \) is bounded on \([-a_{4n}, a_{4n}] \), we may assume that (5.2) holds for all \( n \in \mathbb{N} \). Also by Lemma 2.3 (1),

\[
T(t)/C_1 \leq T(x) \leq C_1T(t) \quad (t \in B_n)
\]

holds true. Let \( g(t) := f(t)\chi_{A_n}(t) \), where \( \chi_A \) is the characteristic function of a set \( A \) and put \( h(t) = f(t) - g(t) \). Since

\[
\int_{\mathbb{R}} \left( \sum_{k=0}^{m-1} p_k^{(j)}(x)p_k(t) \right)^2 w(t)^2 dt = \sum_{k=0}^{m-1} (p_k^{(j)}(x))^2,
\]
(3.2), (5.2) and the Schwarz inequality show that

\[ |s_m^{(j)}(g)(x)w(x)| \leq w(x) \int_{\mathbb{R}} \left| g(t) \sum_{k=0}^{m-1} p_k^{(j)}(x)p_k(t)w(t)^2 \right| dt \]

\[ \leq \left( \sum_{k=0}^{m-1} (p_k^{(j)}(x))^2 w(x)^2 \right)^{1/2} \left( \int_{\mathbb{R}_n} |f(t)w(t)|^2 dt \right)^{1/2} \]

\[ \leq C_2^{(2j+1)/4} \left( \sum_{k=0}^{m-1} \frac{w(x)^2}{T(x)^{2j+1/2}} (p_k^{(j)}(x))^2 \right)^{1/2} \left( \int_{\mathbb{R}_n} T(t)^{(2j+1)/4} |f(t)w(t)|^2 dt \right)^{1/2} \]

\[ \leq C \left( \sum_{k=0}^{m-1} \frac{w(x)^2}{T(x)^{2j+1/2}} (p_k^{(j)}(x))^2 \right)^{1/2} \|T^{(2j+1)/4}fw\|_{L^\infty(\mathbb{R})} \left( \frac{\alpha_{2n}}{2n} \right)^{1/2}. \]

Since \( v_n^{(j)}(g)(x) = (1/n) \sum_{m=n+1}^{2n} s_m^{(j)}(g)(x) \), Proposition 3.2 gives us

\[ |v_n^{(j)}(g)(x)w(x)| \leq C \left( \frac{n}{\alpha_n} \right)^j \|T^{(2j+1)/4}fw\|_{L^\infty(\mathbb{R})} \]

for all \( x \in \mathbb{R} \) with \( |x| \leq a_{2n} \).

To estimate \( v_n^{(j)}(h) \), we use (3.3). For \( i = 0, 1, \ldots, j \), we put

\[ v_{n,i}(h)(x) := \frac{1}{n} \sum_{m=n+1}^{2n} \frac{\gamma_{m-1}}{\gamma_m} \int_{\mathbb{R}} h(t) \frac{p_m^{(j-i)}(x)p_m(t) - p_m^{(j-i)}(t)p_m(t)}{(x-t)^{i+1}} w(t)^2 dt \]

\[ = \frac{1}{n} \sum_{m=n+1}^{2n} \frac{\gamma_{m-1}}{\gamma_m} (b_{m-1}(h_i)p_m^{(j-i)}(x) - b_m(h_i)p_m^{(j-i)}(x)), \]

where

\[ h_i(t) := \frac{h(t)}{(x-t)^{i+1}} \quad \text{and} \quad b_k(h_i) := \int_{\mathbb{R}} h_i(t)p_k(t)w(t)^2 dt \quad (k \in \mathbb{N} \cup \{0\}). \]

Then

\[ v_n^{(j)}(h)(x) = \sum_{i=0}^{j} (-1)^i \binom{j}{i} v_{n,i}(h)(x). \]
By (3.4), the Schwarz inequality and Proposition 3.2, we have

\[
|v_{n,i}(h)(x)w(x)| 
\leq \frac{1}{n} \sum_{m=0}^{2n} \gamma_{m-1} \frac{T(p^{(j-i)}_m)(x)b_{m}(h_i)w(x)}{\gamma_m}
\]

\[
\leq C \frac{a_n}{n} \left( w(x)^2 \sum_{m=0}^{2n} (p^{(j-i)}_m(x))^2 \right)^{1/2} \left( \sum_{m=0}^{2n} |b_{m}(h_i)|^2 \right)^{1/2}
\]

\[
\leq C \frac{a_n}{n} \left( \frac{w(x)^2}{(T(x)^{2(j-i)+1/2}} \sum_{m=0}^{2n} (p^{(j-i)}_m(x))^2 \right)^{1/2} \left( T(x)^{2(j-i)+1/2} \sum_{m=0}^{2n} |b_{m}(h_i)|^2 \right)^{1/2}
\]

\[
\leq C \left( \frac{n}{a_n} \right)^{(2(j-i)-1)/2} \left( T(x)^{2(j-i)+1/2} \sum_{m=0}^{2n} |b_{m}(h_i)|^2 \right)^{1/2}.
\]

The Bessel inequality implies that

\[
\sum_{m=0}^{2n} |b_{m}(h_i)|^2 \leq \int_{\mathbb{R}} \frac{h(t)}{(x-t)^{i+1}} w(t)dt = \int_{B_n \cap C_n} \frac{|f(t)w(t)|^2}{(x-t)^{2(i+1)}} dt
\]

and hence, by (5.3), we have

\[
T(x)^{2(j-i)+1/2} \int_{B_n} \frac{|f(t)w(t)|^2}{(x-t)^{2(i+1)}} dt
\]

\[
\leq C_1^{(2(j-i)+1/2)} \int_{B_n} \frac{|T(t)^{(2(j-i)+1/4)}f(t)w(t)|^2}{(x-t)^{2(i+1)}} dt
\]

\[
\leq C \|T^{(2(j-i)+1/4)}f w\|_{L^\infty(\mathbb{R})} \int_{|x-t| > \frac{a_{2n}}{2}} \frac{1}{(x-t)^{2(i+1)}} dt
\]

\[
\leq C \|T^{(2(j-i)+1/4)}f w\|_{L^2(\mathbb{R})} \left( \frac{n}{a_n} \right)^{2i+1}
\]

\[
\leq C \|T^{(2(j-i)+1/4)}f w\|_{L^2(\mathbb{R})} \left( \frac{n}{a_n} \right)^{2i+1},
\]

because \( T \geq 1 \). On the other hand, if \( |x| \leq a_{2n} \) then \( T(x) \leq CT(a_n) \), so that

\[
T(x)^{2(j-i)+1/2} \int_{C_n} \frac{|f(t)w(t)|^2}{(x-t)^{2(i+1)}} dt
\]

\[
\leq C \|w\|_{L^\infty(\mathbb{R})} T(x)^{2(j-i)+1/2} \int_{C_n} \frac{1}{(x-t)^{2(i+1)}} dt
\]

\[
\leq C \|w\|_{L^2(\mathbb{R})} \int_{T(x)^{2(j-i)+3/2}} T(x)^{(2(j-i)+3/2)}
\]

\[
\leq C \|T^{(2(j-i)+1/4)}f w\|_{L^2(\mathbb{R})} T(a_n)^{(2(j-i)+3/2)}.
\]
Moreover
\[ (5.6) \quad T(a_n)^{(2(i+k)+3)/2} \leq C \left( \frac{n}{a_n} \right)^{2i+1} \]
holds. In fact, to show this we may assume that \( w \) is an Erdős-type weight by (1.4). Then by Lemma 2.1 (5) and Lemma 2.6, we have
\[ T(a_n)^{(2k+3)/2} \leq C n^{(2/(2k+3) - \delta)((2k+3)/2)} \leq C n^{1-\delta'} \leq C \left( \frac{n}{a_n} \right). \]
Similarly
\[ T(x)^{(2(i+k)+3)/2} \leq C T(a_n)^{(4k+3)/2} \leq C n^{(2/(2k+3) - \delta)((4k+3)/2)} \leq C n^{1-\delta''} \leq C \left( \frac{n}{a_n} \right)^{2i+1} \]
holds for \( i \geq 1 \). Combining the above estimates, we thus have
\[
\begin{align*}
|v_{n,i}(h)(x)w(x)| &\leq C \left( \frac{n}{a_n} \right)^{(2(j-i)-1)/2} \left( T(x)^{(2(j-i)+1)/2} \sum_{m=0}^{2n} |b_m(h)|^2 \right)^{1/2} \\
&\leq C \left( \frac{n}{a_n} \right)^{(2(j-i)-1)/2} ||T^{(2j+1)/4} f w||_{L^\infty(\mathbb{R})} \left( \frac{n}{a_n} \right)^{(2i+1)/2} \\
&\leq C \left( \frac{n}{a_n} \right)^j ||T^{(2j+1)/4} f w||_{L^\infty(\mathbb{R})}.
\end{align*}
\]
It follows from (5.5) that
\[
|v^{(j)}_n(h)(x)w(x)| \leq C \left( \frac{n}{a_n} \right)^j ||T^{(2j+1)/4} f w||_{L^\infty(\mathbb{R})}.
\]
This together with (5.4) shows (5.1).

We will prove (1.13) for \( p = 2 \) in the next section. Then using the Riesz-Thorin interpolation theorem for an operator

\[ F : f \mapsto \mathcal{W}^{(j)}_n \left( \frac{f}{w} \right), \]

we obtain (1.13) for all \( 2 \leq p \leq \infty \). This completes the proof of Theorem 1.2.

6. Proof of Theorem 1.3

Let \( 1 \leq p \leq 2 \) and \( T^{(2j+1)/4} f w \in L^2(\mathbb{R}) \). We use the same notations as in the previous section. Then as in the estimate of \( s^{(j)}_m(g) \) in the previous section, we have
\[ (6.1) \quad |s^{(j)}_m(g)(x)w(x)| \leq C \left( \frac{n}{a_n} \right)^{(2j+1)/2} \left( \int_{A_n} |T(t)^{(2j+1)/4} f(t)w(t)|^2 dt \right)^{1/2} \]
for $|x| \leq a_{2n}$. Hence Lemma 2.2 and the Hölder inequality imply

\[
\int_{\mathbb{R}} |s_{m}^{(j)}(g)(x)w(x)|^p dx \leq 2^p \int_{|x| \leq a_{2n}} |s_{m}^{(j)}(g)(x)w(x)|^p dx \\
\leq C \int_{|x| \leq a_{2n}} \left( \frac{n}{a_n} \right)^{p(2j+1)/2} \left( \int_{A_n} |T(t)^{(2j+1)/4}f(t)w(t)|^2 dt \right)^{p/2} dx \\
\leq C \left( \frac{n}{a_n} \right)^{p(2j+1)/2} \int_{|x| \leq a_{2n}} \left( \int_{|u| \leq \frac{a_{2n}}{a_n}} |T(x-u)^{(2j+1)/4}f(x-u)w(x-u)|^2 du \right)^{p/2} dx \\
\leq C \left( \frac{n}{a_n} \right)^{p(2j+1)/2} a_n^{(2-p)/2} \\
\times \left\{ \int_{|x| \leq a_{2n}} \left( \int_{|u| \leq \frac{a_{2n}}{a_n}} |T(x-u)^{(2j+1)/4}f(x-u)w(x-u)|^2 du \right) dx \right\}^{p/2} \\
\leq C \left( \frac{n}{a_n} \right)^{p(2j+1)/2} a_n^{(2-p)/2} \left\| T^{(2j+1)/4}f w \right\|_{L^2(\mathbb{R})}^{p/2}.
\]

do that we have

\begin{equation}
\|v_{n}^{(j)}(g)w\|_{L^p(\mathbb{R})} \leq C \left( \frac{n}{a_n} \right)^j a_n^{(2-p)/(2p)} \left\| T^{(2j+1)/4}f w \right\|_{L^2(\mathbb{R})}.
\end{equation}

Next we estimate $v_{n,i}(h)$. Similarly as above, we have

\[
\int_{\mathbb{R}} |v_{n,i}(h)(x)w(x)|^p dx \leq 2 \int_{|x| \leq a_{2n}} |v_{n,i}(h)(x)w(x)|^p dx \\
\leq C \left( \frac{n}{a_n} \right)^{p(2j-i-1)/2} a_n^{(2-p)/2} \\
\times \left\{ \int_{|x| \leq a_{2n}} \left( \int_{B_n \cup C_n} \frac{|T(t)^{(2j-i+1)/4}f(t)w(t)|^2}{(x-t)^{2(i+1)}} dt \right) dx \right\}^{p/2}.
\]

Also as in the argument of previous section,

\[
\int_{|x| \leq a_{2n}} \left( \int_{B_n} \frac{|T^{(2j-i+1)/4}(t)f(t)w(t)|^2}{(x-t)^{2(i+1)}} dt \right) dx \\
\leq \int_{\mathbb{R}} \left( \int_{|u| \leq \frac{a_{2n}}{a_n}} \frac{|T^{(2j-i+1)/4}(x-u)f(x-u)w(x-u)|^2}{u^{2(i+1)}} du \right) dx \\
\leq C \left( \frac{n}{a_n} \right)^{2i+1} \|T^{(2j+1)/4}f w\|_{L^2(\mathbb{R})}^2 \leq C \left( \frac{n}{a_n} \right)^{2i+1} \|T^{(2j+1)/4}f w\|_{L^2(\mathbb{R})}^2.
\]
On the other hand, by (5.6) we have
\[
\int_{|x| \leq a_n} \left( \frac{T(t)^2 (j-i+1)}{2} \right)^{1/2} \left( \frac{|f(t)w(t)|}{|x|^{2(j-i+1)}} dt \right) dx
\]
\[
\leq C T(a_{2n})^{2(j-i+1)/2} \int_{\mathbb{R}} \left( \int_{|t| \leq |u|} \frac{|f(x-u)w(x-u)|}{u^{2(j-i+1)}} du \right) dx
\]
\[
\leq C \|w\|_{L^2(\mathbb{R})}^2 T(a_{2n})^{2(j-i+1)/2} \int_{\mathbb{R}} \frac{1}{u^{2(j-i+1)}} du
\]
\[
\leq C T(a_{2n})^{2(j-i+1)/2} \|w\|_{L^2(\mathbb{R})}^2 \]

Consequently we have
\[
(6.3) \quad \|v_{n,i}(h)w\|_{L^p(\mathbb{R})} \leq C \left( \frac{n}{a_n} \right)^j a_n^{(2-p)/2p} \|T(2j+1)/4 f w\|_{L^2(\mathbb{R})}
\]
for \(0 \leq i \leq j\), so that
\[
\|v_{n}^{(j)}(h)w\|_{L^p(\mathbb{R})} \leq C \left( \frac{n}{a_n} \right)^j a_n^{(2-p)/2p} \|T(2j+1)/4 f w\|_{L^2(\mathbb{R})}
\]
follows. This together with (6.2) shows (1.14). This completes the proof of Theorem 1.3.

Under the same assumptions in Theorem 1.3, the following estimate is also established. Let \(\beta > 1\) and \(1 \leq p \leq 2\). Then
\[
(6.4) \quad \|v_{n}^{(j)}(f)\|_{L^p(\mathbb{R})} \leq C \left( \frac{n}{a_n} \right)^j \|T(2j+1)/4 f w\|_{L^2(\mathbb{R})}
\]
holds for every \(T(2j+1)/4 f w \in L^2(\mathbb{R})\) and every \(n \in \mathbb{N}\). In fact, in the proof of Theorem 1.3, we used
\[
\int_{|x| \leq a_n} \left( \int_{|t| \leq \frac{a_n}{2n}} |T(t)^{2j+1}/4 f(t)w(t)|^2 dt \right)^{p/2} dx
\]
\[
\leq a_n^{(2-p)/2} \left\{ \int_{|x| \leq a_n} \left( \int_{|t| \leq \frac{a_n}{2n}} |T(t)^{2j+1}/4 f(t)w(t)|^2 du \right) dx \right\}^{p/2},
\]
which follows from the H"older inequality. Instead of this, we use
\[
\int_{\mathbb{R}} \left( \frac{1}{1 + |x|^{(2-p)/2}} \right)^{(2-p)/2} \left\{ \int_{|t| \leq \frac{a_n}{2n}} |T(t)^{2j+1}/4 f(t)w(t)|^2 dt \right\}^{p/2} dx
\]
\[
\leq \left( \int_{\mathbb{R}} \left( \frac{1}{1 + |x|^{(2-p)/2}} \right)^{(2-p)/2} \left\{ \int_{|t| \leq \frac{a_n}{2n}} |T(t)^{2j+1}/4 f(t)w(t)|^2 dt \right\}^{p/2} dx \right)^{1/(2-p)/2}.
\]
Then as in (6.2), we obtain

$$\|v_{n}^{(j)}(g)\|_{L^p(R)} \leq C \left( \frac{n}{a_n} \right)^j \|T^{(2j+1)}/4fw\|_{L^2(R)}.$$ 

For the estimate of $v_{n, i}(h)$, we take $w^* \in F_{\chi}(C^3+)$ such that $w^*(x) \sim w(x)/(1 + |x|)^{(2-p)\beta/(2p)}$ (see [5, Theorem 4.2]). Then by Lemma 2.2,

$$\int_{R} \left| v_{n, i}(h) \frac{w(x)}{(1 + |x|)^{(2-p)\beta/(2p)}} \right|^p dx \leq 2^p \int_{|x| \leq a_n^2} \left| v_{n, i}(h) \frac{w(x)}{(1 + |x|)^{(2-p)\beta/(2p)}} \right|^p dx.$$

By an estimate similar to (6.3), we obtain

$$\|v_{n, i}(h)\|_{L^p(R)} \leq C \left( \frac{n}{a_n} \right)^j \|T^{(2j+1)}/4fw\|_{L^2(R)},$$

which shows (6.4).

References


